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Fourier series and its Physical Application-A Study

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Abstract

Fourier series are of great importance in both theoretical and applied mathematics. This paper will focus on the Fourier series of the complex exponentials of the many possible methods of estimating complex valued functions, Fourier series are especially attractive because uniform convergence of the Fourier series (as more terms are added) is guaranteed for continuous , bounded functions. After studying this paper we will learn about how Fourier transforms is useful in many physical applications, such as partial differential equations and heat transfer equations. one can solve many important problems of physics with very simple way.

Key words : Fourier Transform, Fourier Sine and Cosine Transform, Fourier Differential equation.

Introduction

The central starting point of Fourier analysis is Fourier series[1]. They are infinite series designed to represent general periodic functions in terms of simple ones, namely, cosines and sines. This trigonometric system is orthogonal, allowing the computation of the coefficients of the Fourier series by use of the well-known Euler formulas, Fourier series are very important to the engineer and physicist because they allow the solution of linear differential equations and partial differential equations. Fourier series are, in a certain sense, more universal than the familiar Taylor series in calculus because many discontinuous periodic functions that come up in applications can be developed in Fourier series but do not have Taylor series expansions. The Fourier Transform is a tool that breaks a waveform (a function or signal) into an alternate representation, characterized by sine and cosines. The Fourier Transform shows that any waveform can be rewritten as the sum of sinusoidal functions. The Fourier transform is a mathematical function that decomposes a waveform, which is a function of time, into the frequencies that make it up. The result produced by the Fourier transform is a complex valued function of frequency. The absolute value of the Fourier transform represents the frequency value present in the original function and its complex argument represents the phase offset of the basic sinusoidal in that frequency. The Fourier transform is also called a generalization of the Fourier series. This term can also be applied to both the frequency domain representation and the mathematical function used. The Fourier transform helps in extending the Fourier series to non-periodic functions, which allows viewing any function as a sum of simple sinusoids.

Fourier series :

A Fourier series[1] is an expansion of a periodic function in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonally relationships of the sine and cosine functions. The computation and study of Fourier series is known as harmonic analysis and is extremely useful as a way to break up an *arbitrary* periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical. Examples of successive approximations to common functions using Fourier series are illustrated above.

In particular, since the superposition principle holds for solutions of a linear homogeneous ordinary differential equation, if such an equation can be solved in the case of a single sinusoid, the solution for an arbitrary function is immediately available by expressing the original function as a Fourier series and then plugging in the solution for each sinusoidal component. In some special cases where the Fourier series can be summed in closed form, this technique can even yield analytic solutions.

Any set of functions that form a complete orthogonal system have a corresponding generalized Fourier series analogous to the Fourier series.

The Fourier Series, the founding principle behind the field of Fourier Analysis, is an infinite expansion of a function in terms of sines and cosines or imaginary exponentials. The series is defined in its imaginary exponential form as follows:

$$f(t) = \sum_{n=-\infty}^{n=+\infty} a_n \, e^{inx} \tag{1}$$

where the a_n's are given by the expression

$$\boldsymbol{a_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x) e^{-inx}\, dx} \tag{2}$$

Thus, the Fourier Series is an infinite superposition of imaginary exponentials with frequency terms that increase as n increases. Since sines and cosines (and in turn, imaginary exponentials) form an orthogonal set[1], this series converges for any moderately well-behaved function f(x).

For example, using orthogonality of the roots of a Bessel function of the first kind gives a so-called Fourier-Bessel series.

• A function f (x) is called a periodic function if f (x) is defined for all real x, except possibly at some points, and if there is some positive number p, called a period of f (x) such that

$$f\ (x+p) = f\ x\ for\ all\ \ x$$

• Familiar periodic functions are the cosine, sine, tangent, and cotangent. Examples of functions that are not periodic are x, x^2 , x^3 , $e^{\ x}$, cos*h x etc.* to mention just a few.

If f(x) has a period of p then it has also a period of 2p

$$f\ (x+2p)\ = f\{(\ x+p)+p\} = f\ (x+p) = f(x)$$

Or in general we can write (+ *np*) = *f* (x)

•A Fourier series is defined as an expansion of a real function or representation of a real function in a series of sines and cosines such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where a_0, a_n, *and* b_n are constants, called the Fourier coefficients of the series. We see that each term has the period of 2π Hence if the coefficients are such that the series converges, its sum will be a function of period 2π.

The Fourier coefficients of f(x), given by the Euler formulas

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx \qquad n = 1,2,3\ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx \qquad n = 1,2,3\ldots$$

The above Fourier series is given for period 2π. The transition from period 2π to be period *p* = 2*L* is effected by a suitable change of scale, as follows. Let (*x*) have period = 2 . Then we can introduce a new variable v such that, (*x*) as a function of v, has period 2π.

If we set $x = \frac{p}{2\pi} v \rightarrow v = \frac{\pi}{L} x$

This means v = ±π corresponds to x = ± *L* this represent f , as function of v has a period of 2π, hence the Fourier series is

$$f(v) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nv + \sum_{n=1}^{\infty} b_n \sin nxv$$

Now using $v = \frac{\pi}{L}$ x Fourier series for the period of (-L,L) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\frac{\pi}{L}x + \sum_{n=1}^{\infty} b_n \sin n\frac{\pi}{L}x$$

This is Fourier series we obtain for a function of f(x) period of 2L the Fourier series.

The coefficient is given by

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Fourier Transform of Derivatives :

As we know that from the properties of Fourier Transform

$$F\{f^n(x)\} = (-is)^n F(s)$$

$$F\left(\frac{\partial 2f}{dx2}\right) = (-is)^2 F\{f(x)\} = -s^2 \bar{f} \text{ [where } \bar{f} \text{ is Fourier Transform of f]}$$

If F_c and F_s are cosine and sine Fourier transform f(x) then

$$F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}}f(0) + s F_s(s)$$

Proof : From cosine Fourier transform we know that

$$F_c\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos sx\, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos sx\, d\{f(x)\}$$

Now integrating by parts, we get

$$= \sqrt{\frac{2}{\pi}}[\cos sx\, f(x)]_0^\infty - \sqrt{\frac{2}{\pi}}\left[-s\int_0^\infty \sin sx\, f(x)dx\right]$$

$$= \sqrt{\frac{2}{\pi}}[0 - f(0)] + s\sqrt{\frac{2}{\pi}}\left[\int_0^\infty \sin sx\, f(x)dx\right] \quad \{\text{Assuming } f(x) \to 0 \text{ as } x \to \infty\}$$

Hence

$$F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}}f(0) + s F_s(s) \text{ where } \sqrt{\frac{2}{\pi}}\left[\int_0^\infty \sin sx\, f(x)\right] = F_s(s)$$

Fourier Transform of Partial Derivative of a Function :

The Fourier transform of the partial derivatives[2] is given by $F\left[\frac{\partial 2u}{\partial 2x}\right] = -s^2 F(u)$

Where F(u) is the Fourier transform of u

The Fourier sine transform of the partial derivative is given by

$$F_s\left[\frac{\partial 2u}{\partial 2x}\right] = s(u)_{x=0} - s^2 F_s(u)$$

Where $F_s(u)$ is the Fourier Sine transform of u

The Fourier Cosine transform of the partial derivative is given by

$$F_c\left[\frac{\partial 2u}{\partial 2x}\right] = -\left[\frac{\partial 2u}{\partial 2x}\right]_{x=0} - s^2 F_c(u)$$

Where $F_c(u)$ is the Fourier cosine transform of u

Conditions for A Fourier Expansion :

The reader must not be misled by the belief that the Fourier series expansion of f(x) in each case shall be valid. The above discussion has merely shown that if f(x) has an expansion, then the coefficients are given by Euler's formulae. The problems concerning the possibility of expressing a function by Fourier series and convergence of this series are many and cumbersome. Such questions should be left to the curiosity of a pure-mathematician. However, almost all engineering applications are covered by the following well-known Dirichlet's conditions:

Any function f(x) can be developed as a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0, a_n, b_n are constants, provided :

(i)f(x) is periodic, single valued and finite;

(ii)f (x) has finite number of discontinuities in any one period;

(iii)f(x) has at the most a finite number of maxima and minima.

Functions Having Point of Discontinuity :

In deriving the Euler's formulae for a_0, a_n, b_n it was assumed that f(x) was continuous. Instead a function may have a finite number of points of finite discontinuity *i.e.* its graph may consists of a finite number of different curves given by different equations. Even then such a function is expressible as a Fourier series. For instance, if in the interval (α, $\alpha+2\pi$), f(x) is Defined by

$$f(x) = \begin{cases} \emptyset(x), \alpha < x < c \\ \varphi(x), c < x < \alpha + 2\pi \end{cases}$$

ie c is the point of discontinuity, then

$$a_0 = \frac{1}{\pi}\left[\int_a^c \emptyset(x)dx + \int_c^{a+2\pi} \varphi(x)dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_a^c \emptyset(x)cosnx\, dx + \int_c^{a+2\pi} \varphi(x)cosnx\, dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_a^c \emptyset(x)sinnx\, dx + \int_c^{a+2\pi} \varphi(x)sinnx\, dx\right]$$

At a point of finite discontinuity x = c , there is a finite jump in the graph of function. Both the limit on the left f (c^-) and the limit on right f(c^+) exist and different. At such a point, Fourier series gives the value of f (x) as the arithmetic mean of these two limits*, i.e.* at x = c,

$$f(x)= \frac{f(c-)+f(c+)}{2}$$

Partial Differential Equations :

In this topic we are going to take a very brief look at one of the more common methods for solving simple partial differential equations. The method we'll be taking a look at is that of Separation of Variables. We need to make it very clear before we even start this topic that we are going to be doing nothing more than barely scratching the surface of not only partial differential equations but also of the method of separation of variables. It would take several most of the basic techniques for solving partial differential equations. Here is a brief listing of the topics covered in this topic.

The Heat Equation – a partial derivation of the heat equation that can be solved to give the temperature in a one dimensional bar of length L. In addition, we give several possible boundary conditions that can be used in this situation. We also define the Laplacian and give a version of the heat equation for two or three dimensional situations.

Solving the Heat Equation – we go through the complete separation of variables process, including solving the two ordinary differential equations the process generates. We will do this by solving the heat equation with three different sets of boundary conditions. Included is an example solving the heat equation on a bar of length L but instead on a thin circular ring.

Heat Equation with Non-Zero Temperature Boundaries – we take a quick look at solving the heat equation in which the boundary conditions are fixed, non-zero temperature. when we generally required the boundary conditions to be both fixed and zero.

The Heat Equation :

Before we get into actually solving partial differential equations and before we even start discussing the method of separation of variables we want to spend a little bit of time talking about the two main partial differential equations that we'll be solving later on in the chapter. We'll look at the first one in this section and the second one in the next section. The first partial differential equation that we'll be looking at once we get started with solving will be the heat equation, which governs the temperature distribution in an object. We are going to give several forms of the heat equation for reference purposes, but we will only be really solving one of them. We will start out by considering the temperature in a 1-D bar of length L. What this means is that we are going to assume that the bar starts off at x = 0 and ends when we reach x L = . We are also going to so assume that at any location, x the temperature will be constant at every point in the cross section at that x. In other words, temperature will only vary in x and we can hence consider the bar to be a 1-D bar. Note that with this assumption the actual shape of the cross section (i.e. circular, rectangular, etc.) doesn't matter. Note that the 1-D assumption is actually not all that bad of an assumption as it might seem at first glance.

If we assume that the lateral surface of the bar is perfectly insulated (*i.e.* no heat can flow through the lateral surface) then the only way heat can enter or leave the bar as at either end. This means that heat can only flow from left to right or right to left and thus creating a 1-D temperature distribution. The assumption of the lateral surfaces being perfectly insulated is of course impossible, but it is possible to put enough insulation on the lateral surfaces that there will be very little heat flow through them and so, at least for a time, we can consider the lateral surfaces to be perfectly insulated. Okay, let's now get some definitions out of the way before we write down the first form of the heat equation.

u(x, t) = Temperature at any point x and any time t

c(x) = Specific Heat

ρ(x) = Mass Density

φ(x, t) = Heat Flux

Q(x, t) = Heat energy generated per unit volume per unit time

We should probably make a couple of comments about some of these quantities before proceeding. The specific heat, c (x) > 0, of a material is the amount of heat energy that it takes to raise one unit of mass of the material by one unit of temperature. As indicated we are going to assume, at least initially, that the specific heat may not be uniform throughout the bar. Note as well that in practice the specific heat depends upon the temperature. However, this will generally only be an issue for large temperature differences (which in turn depends on the material the bar is made out of) and so we're going to assume for the purposes of this discussion that the temperature differences are not large enough to affect our solution. The mass density, ρ(x), is the mass per unit volume of the material. As with the specific heat we're going to initially assume that the mass density may not be uniform throughout the bar. The heat flux, φ(x, t), is the amount of thermal energy that flows to the right per unit surface area per unit time. The "flows to the right" bit simply tells us that if φ(x, t) > 0 for some x and t then the heat is flowing to the right at that point and time. Likewise, if φ (x, t) < 0 then the heat will be flowing to the left at that point and time. The final quantity we defined above is Q(x, t) and this is used to represent any external sources or sinks (*i.e.* heat energy taken out of the system) of heat energy. If Q(x, t) > 0 then heat energy is being added to the system at that location and time and if Q(x, t) < 0 then heat energy is being removed from the system at that location and time. With these quantities the heat equation is,

$$c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \varphi}{\partial x} + Q(x, t) \quad (1)$$

While this is a nice form of the heat equation it is not actually something we can solve. In this form there are two unknown functions, u and φ , and so we need to get rid of one of them. With Fourier's law we can easily remove the heat flux from this equation.

Fourier's law states that, $\varphi(x,t) = -K_0\,(x)\dfrac{\partial u}{\partial x}$

where $K_0(x) > 0$ is the thermal conductivity of the material and measures the ability of a given material to conduct heat. The better a material can conduct heat the larger $K_0(x)$ will be. As noted the thermal conductivity can vary with the location in the bar. Also, much like the specific heat the thermal conductivity can vary with temperature, but we will assume that the total temperature change is not so great that this will be an issue and so we will assume for the purposes here that the thermal conductivity will not vary with temperature. Fourier's law does a very good job of modelling what we know to be true about heat flow. First, we know that if the temperature in a region is constant, *i.e.* $\frac{\partial u}{\partial x} = 0$ then there is no heat flow.

Next, we know that if there is a temperature difference in a region we know the heat will flow from the hot portion to the cold portion of the region. For example, if it is hotter to the right then we know that the heat should flow to the left. When it is hotter to the right then we also know that $\frac{\partial u}{\partial x} = 0 > 0$ (*i.e.* the temperature increases as we move to the right) and so we'll have j < 0 and so the heat will flow to the left as it should. Likewise if $\frac{\partial u}{\partial x} < 0$ (*i.e.* it is hotter to the left) then we'll have j > 0 and heat will flow to the right as it should. Finally, the greater the temperature difference in a region (ie the larger is $\frac{\partial u}{\partial x}$) then the greater the heat flow. So, if we plug Fourier's law into (1), we get the following form of the heat equation,

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + Q(x, t) \qquad (2)$$

Note that we factored the minus sign out of the derivative to cancel against the minus sign that was already there. We cannot however, factor the thermal conductivity out of the derivative since it is a function of x and the derivative is with respect to x.

Solving (2) is quite difficult due to the non uniform nature of the thermal properties and the mass density. So, let's now assume that these properties are all constant, i.e.,

$c(x) = c \qquad \rho(x) = \rho \qquad K_0(x) = K_0$

where c, ρ and K_0 are now all fixed quantities. In this case we generally say that the material in the bar is uniform. Under these assumptions the heat equation becomes,

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial 2u}{\partial x2} + Q(x, t) \qquad (3)$$

For a final simplification to the heat equation let's divide both sides by cr and define the thermal diffusivity to be, $k = \frac{K_0}{c\rho}$

The heat equation is then $$\frac{\partial u}{\partial x} = k\,\frac{\partial 2u}{\partial x2} + \frac{Q(x,t)}{c\rho} \qquad (4)$$

To most people this is what they mean when they talk about the heat equation and in fact it will be the equation that we'll be solving. Well, actually we'll be solving (4) with no external sources, *i.e*. Q(x, t) = 0 , but we'll be considering this form when we start discussing separation of variables in a couple of sections. We'll only drop the sources term when we actually start solving the heat equation. Now that we've got the 1-D heat equation taken care of we need to move into the initial and boundary conditions we'll also need in order to solve the problem. If you go back to any of our solutions of ordinary differential equations that we've done in previous sections you can see that the number of conditions required always matched the highest order of the derivative in the equation. In partial differential equations the same idea holds except now we have to pay attention to the variable we're differentiating with respect to as well. So, for the heat equation we've got a first order time derivative and so we'll need one initial condition and a second order spatial derivative and so we'll need two boundary conditions. The initial condition that we'll use here is, $u(x,0) = f(x)$ and we don't really need to say much about it here other than to note that this just tells us what the initial temperature distribution in the bar is. The boundary conditions will tell us something about what the temperature and/or heat flow is doing at the boundaries of the bar. There are four of them that are fairly common boundary conditions.

The first type of boundary conditions that we can have would be the prescribed temperature boundary conditions, also called Dirichlet conditions. The prescribed temperature boundary conditions are $u(0,t)= g_1(t)$ $u(L, t)= g_2(t)$

The next type of boundary conditions are prescribed heat flux, also called Neumann conditions. Using Fourier's law these can be written as,

$$-K(0)\frac{\partial u}{\partial x}(0,t) = \varphi_1(t) \qquad -K(L)\frac{\partial u}{\partial x}(L,t) = \varphi_2(t)$$

If either of the boundaries are perfectly insulated, *i.e.* there is no heat flow out of them then these boundary conditions reduce to, $\frac{\partial u}{\partial x}(0,t) = 0$ $\frac{\partial u}{\partial x}(L,t) = 0$

and note that we will often just call these particular boundary conditions insulated boundaries and drop the "perfectly" part.

The third type of boundary conditions use Newton's law of cooling and are sometimes called Robins conditions. These are usually used when the bar is in a moving fluid and note we can consider air to be a fluid for this purpose.

Here are the equations for this kind of boundary condition.

$$-K(0)\frac{\partial u}{\partial x}(0,t) = H[u(0,t) - g_1(t)] \qquad -K(L)\frac{\partial u}{\partial x}(L,t) = H[u(L,t) - g_2(t)]$$

where H is a positive quantity that is experimentally determined and $g_1(t)$ and $g_2(t)$ give the temperature of the surrounding fluid at the respective boundaries. Note that the two conditions do vary slightly depending on which boundary we are at. At $x = 0$ we have a minus sign on the right side while we don't at $x = L$. To see why this is let's first assume that at $x = 0$ we have $u(0,t) > g_1(t)$. In other words, the bar is hotter than the surrounding fluid and so at $x = 0$ the heat flow (as given by the left side of the equation) must be to the left, or negative since the heat will flow from the hotter bar into the cooler surrounding liquid. If the heat flow is negative then we need to have a minus sign on the right side of the equation to make sure that it has the proper sign.

If the bar is cooler than the surrounding fluid at $x = 0$, *i.e.* $u(0, t) < g_1(t)$ we can make a similar argument to justify the minus sign. We'll leave it to you to verify this. If we now look at the other end, $x = L$, and again assume that the bar is hotter than the surrounding fluid or, $u(L, t) > g_2(t)$. In this case the heat flow must be to the right, or be positive, and so in this case we can't have a minus sign. Finally, we'll again leave it to you to verify that we can't have the minus sign at $x = L$ is the bar is cooler than the surrounding fluid as well.

Physical Applications of Fourier series :

1. Spectrum Analyzer :

An important instrument to any experimentalist is the spectrum analyser[7]. This instrument reads a signal (usually a voltage) and provides the operator with the Fourier coefficients which correspond to each of the sine and cosine terms of the Fourier expansion of the signal. Suppose an instrument takes a time-domain signal, such as the amplitude of the output voltage of an instrument. Thus a digital oscilloscope that is sufficiently fast and equipped with a FFT algorithm is capable of providing the user with the frequency components of the source signal. Oscilloscopes which are equipped with the ability to FFT their inputs are termed "Digital Spectral Analyzers". Although they were once a separate piece of equipment for experimentalists, improvements in digital electronics has made it practical to merge the role of oscilloscopes with that of the Spectral Analyzer; it is quite common now that FFT algorithms come built into oscilloscopes. Spectrum Analyzers have many uses in the laboratory, but one of the most common uses is for signal noise studies. the FFT of the signal gives the amplitudes of the various oscillatory components of the input. After normalization, this allows for the experimentalist to determine what frequencies dominate their signal. For example, if we have a DC signal, we would expect the FFT to show only very low frequency oscillations (i.e., the largest amplitudes should correspond to $f \approx 0$). However, if we see a sharp peak of amplitudes around 60 Hz, we would know that something is feeding noise into our signal with a frequency of 60 Hz (for example, an AC leakage from our power source).

2. Digital Signal Processing :

We have already seen how the Fourier series allows experimentalists to identify sources of noise. It may also be used to eliminate sources of noise by introducing the idea of the Inverse Fast Fourier Transform (IFFT)[8]. In general, the goal of an Inverse Fourier Transform is to take (the ones that appear and use them to reconstruct the original function, f(t). Analytically, this is done by multiplying each A_n by $e^{2\pi ik\, n/N}$ then taking the sum over all n. However, this is an inefficient algorithm to use when the calculation must be done numerically. Just as there is a fast numerical algorithm for approximating the Fourier coefficients (the FFT), there is another efficient algorithm, called the IFFT, which is capable of calculating the Inverse Fourier Transform much faster than the brute-force method. In 1988, it was shown by Duhamel, Piron , and Etcheto[8] that the IFFT is simply

$$F^{-1}(x) = F(ix^*)^* \qquad \text{(a)}$$

In other words, we can calculate the IFFT directly from the FFT; we simply flip the real and imaginary parts of the coefficients calculated by the original FFT. Thus, the IFFT algorithms are essentially the same as the FFT algorithms; all one must do is flip the numbers around at the beginning of the calculation. Since the IFFT inherits all of the speed benefits of the FFT, it is also quite practical to use it in real time in the laboratory. One of the most common applications of the IFFT in the laboratory is to provide Digital Signal Processing (DSP). In general, the idea of DSP is to use configurable digital electronics to clean up, transform, or amplify a signal by first FFT' ing the signal, removing, shifting or damping the unwanted frequency components, and then transforming the signal back using the IFFT on the filtered signal. There are many advantages to doing DSP as opposed to doing analog signal processing. To begin with, practically speaking, we can have a much more complicated filtering function (the function that transforms the coefficients of the DFT[8]) with DSP than analog signal processing. While it is fairly easy to make a single band pass, low pass, or high pass filter with capacitors, resistors, and inductors, it is relatively difficult and time consuming to implement anything more complicated than these three simple filters. Furthermore, even if a more complicated filter was implemented with analog electronics, it is difficult to make even small modifications to the filter (there are exceptions to this, such as FPGA's, but those are also more difficult to implement than simple software solution). DSP is not limited by either of these effects since the processing is (usually) done in software, which can be programmed to do whatever the user desires. Probably the most important advantage that DSP has over analog signal processing is the fact that the processing may be done after the signal has been taken. In modern-day experiments, raw data is often recorded during the experiment and corrected for noise in software during the analysis step. If one filters the signal beforehand (with analog signal processing), it is possible that later in the experiment, the experimenter could find that they filtered out good signal. The only option in this case is for the experiment to be rerun. On the other hand, if the signal processing was done digitally, all that the experimenter has to do is edit their analysis code and rerun the analysis; this could save both time and money.

3. Analytical Applications :

The Fourier series also has many applications in mathematical analysis[6]. Since it is a sum of multiple sines and cosines, it is easily differentiated and integrated[12], which often simplifies analysis of functions such as saw waves which are common signals in experimentation.

A. Discontinuous Functions :

The Fourier series also offers a simplified analytical approach to dealing with discontinuous functions. In other words, nearly every function encountered in physics, both continuous and discontinuous, may be represented in terms of the Fourier series[17]. This gives the Fourier series a distinct advantage over the Taylor Series expansion of a function, since the Taylor Series places much more stringent limits on convergence than the Fourier series does (continuity is a requirement).

B. Convolutions :

The Convolution Theorem states the following:

$$F^{-1}(F[f]F[g]) = f * g \qquad \text{(b)}$$

where F[f] denotes the Fourier Transform of the function f. Since the Fourier Transform may be approximated by a Fourier Series, FFT algorithms may be applied to the numerical calculation of the convolution; in fact, the FFT method is the preferred method of calculating convolutions which prevents the need for direct integration[5].

C. Generalized Fourier Series :

The concept of the Fourier Series may be generalized to any complete orthogonal system of functions. An "orthogonal system" satisfies the following relation[12].

$$\int_R \emptyset_m(x)\emptyset_n(x)w(x)\,dx = c_{m\delta_{mn}} \qquad \text{(a)}$$

In a generalized Fourier Series, we use these functions φ(x) as the expansion functions instead of sines and cosines (or imaginary exponentials). Then our expansion takes on the following form

$$\mathrm{f(x)} = \sum_{n=0}^{n=\infty} a_n \emptyset_n(x) \qquad \text{(b)}$$

One may then find the coefficients an in an analogous way that one finds the coefficients in the Fourier Series[12].

$$\int_R f(x)\emptyset_n(x)w(x)\,dx = c_{m\delta_{mn}} = \int_R \sum_{n=0}^{\infty} a_n \emptyset_m(x)\emptyset_n(x)w(x)\,dx$$

$$= \sum_{n=0}^{n=\infty} a_n \int_R \emptyset_m(x)\emptyset_n(x)w(x)\,d$$

$$= \sum_{n=0}^{\infty} a_n c_m \delta_{mn}$$

where c_n is the normalization constant given by the orthogonality relationship defined in eqation (a). Equating the first and last parts leaves us with

$$\frac{1}{c_n}\int_R f(x)\emptyset_n(x)w(x)\,dx = a_n \qquad \text{(c)}$$

This result is analogous to the result that was presented in equation (2), and can be used to derive these expressions. An example of another complete orthogonal system which can be used as the basis element for a Generalized Fourier Series is the set of Spherical Harmonics. The Spherical Harmonics provide an series that is analogous to the Fourier Series, called the Laplace series, which is given by the expression

$$f(\theta,\emptyset) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} A_l^m\, Y_l^m(\theta,\emptyset) \qquad \text{(d)}$$

Functional expansions of this form are termed "Generalized Fourier Series" since they utilize the orthogonality relationships of functional systems in the same way that the Fourier Series does.

Conclusion

The Fourier Transform is a tool that breaks a waveform (a function or signal) into an alternate representation, characterized by sine and cosines. it shows that any waveform can be rewritten as the sum of sinusoidal functions and is a mathematical function that decomposes a waveform, which is a function of time, into the frequencies that make it up. The absolute value of the it presents the frequency value present in the original function and its complex argument represents the phase offset of the basic sinusoidal in that frequency.it helps in extending the Fourier series to non-periodic functions, which allows viewing any function as a sum of simple sinusoids. it is useful in many applications ranging from experimental instruments to rigorous mathematical analysis techniques. Thanks to modern developments in digital electronics, coupled with numerical algorithms such as the FFT, the Fourier Series has become one of the most widely used and useful mathematical tools available to any scientist.

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References

1. Wolfram, Eric W. "Fourier Series (eq.30)". Math World- -A Wolfram Web Resource. Retrieved 3 November (2021).
2. Cheever, Erik. "Derivation of Fourier Series". lpsa.swarthmore.edu. Retrieved 3 November (2021).
3. Kassel, Germany: University of Kassel. Alpers, B., Mathematics as a service subject at the tertiary level. A state-of-the-art report for the Mathematics Interest Group. Brussel, Belgium: European Society for Engineering Education (SEFI) (2020).
4. Zhengyang Shen, Lingshen He, Zhouchen Lin, and Jinwen Ma. Pdo-econvs: Partial differential operator based equivariant convolutions. In International Conference on Machine Learning, pages 8697–8706. PMLR, (2020).
5. Numerical Methods in C: The Art of Scientific Computing. 2nd Edition. W. H. Press, S. A. Teukolsky, W T. Vetterling, B. P. Flannery. Ch. 20, Section 6. (915-925).
6. Carvalho, P., & Oliveira, P., Mathematics or mathematics for engineering? In Proceedings from 2018 3rd international conference of the Portuguese society for engineering education (CISPEE). Retrievedfrom,https://ieeexplore.ieee.org/document/8593463/authors#authors (2018).
7. Hegde, U. S, Uma S, Aravind P. N and Malashri S, "Fourier Transformation and its Application in Engineering Field", International Journal of innovative Research in Science, Engineering and Technology, Vol. 6, Issue. 6, June (2017).
8. Duhamel, P.; Piron, B.; Etcheto, J.M., "On computing the inverse DFT," Acoustics, Speech and Signal Processing, IEEE Transactions on, vol. 36, no.2, pp.285-286, Feb 1988 URL: http://ieeexplore.ieee.org.proxy. lib.utk.edu:90/iel1/29/98/00001519.pdf
9. Alpers, B., Differences between the usage of mathematical concepts in engineering statics and engineering mathematics education. In R. Göller, R. Biehler, R. Hochmuth, & H.-G. Rück (Eds.), Didactics of mathematics in higher education as a scientific discipline. Conference proceedings (pp. 137–141) (2017).
10. Hande, K. and FarhaVanu, "Application of Fourier series in communication system", International Journal of Scientific and Engineering Research, Vol. 6, Issue 12, Dec- 2015, pp 24- 25.
11. Kaur, B., Sonia Sharma and Prince Verma, "Methodology of Multimedia and Visualization",

International Journal of Computer Application, Vol. 74, No. 16, July 2013, pp. 0975- 8887.
12. Weisstein, Eric W. "Generalized Fourier Series." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/GeneralizedFourierSeries.htm
13. G. Jacobsen, Noise in digital optical transmission system, The Aetech House Library, London, (1994).
14. Y. F Wang application of diffusion processes in Robotics, optical communications and polymer science, Ph.D. Dissertation, Johns Hopkins University, (2001).
15. Weisstein, Eric W. "Generalize Fourier series" From Math World-A wolfram Web Resource. http://mathworld.wolfram web Resource.
16. Walter Rudin. Principles of Mathematical Analysis, Third Edition McGrawHill International Editions (1976).
17. Eric W. Weisstein. "Fourier Series." From Mathworld–a Wolfram Web Resource. http://mathworld.wolfram.com/FourierSeries.html
18. Eric W. Weisstein. "Fourier Transforms." From Mathworld a Wolfram Web Resource. http://mathworld.wolfram.com/FourierTransforms.html