ISSN 2231-346X (Print)
JUSPS-A Vol. 33(7), 84-93 (2021). Periodicity-Monthly
ISSN 2319-8044 (Online)
Section A
JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES
An International Open Free Access Peer Reviewed Research Journal of Physical Sciences
website:- www.ultrascientist.org
Estd. 1989

Mathematical Modelling to Study Effect of Vaccination on Transmission of CORONA Virus

DHRUTI K. PATEL

Lecturer in Mathematics Government Polytechnic, Himmatnagar-383001 (India)
Corresponding Author Email: dkpatel1888@gmail.com
http://dx.doi.org/10.22147/jusps-A/330701

Acceptance Date 19th November, 2021, **Online Publication Date 24th November, 2021**

Abstract

Since 2019 end, whole of the world is fighting for survival against Covid-19. To overcome the pandemic, global pharmaceutical sector started vaccine research. Early 2021, rose with a hope of vaccine discovery and few companies across the globe have invented and started manufacturing Covid-19 vaccine. As on date vaccination is playing a crucial role in curtaining the spread of this deadly virus caused disease. In this paper, a Compartmental Model is developed to study the spread of Covid-19 taking two different categories of human population into consideration. One is the vaccinated population and other is population without vaccination. Expressions for Reproduction Number are derived for Disease Free Equilibrium (DFE) and Endemic Equilibrium. Stability of the equilibria is also discussed.

Key words : Covid-19, Disease transmission model, Basic reproduction number, Eigen Values.

Classification Codes: 34A34, 34D23, 92-10

1 Introduction

In December 2019, the first case of an unidentified pneumonia was found in Wuhan, China. Later on the responsible virus was named as novel corona virus and the disease is known as Covid-19 [1]. The disease has spread to almost 210 countries and the current scenario is declared as pandemic by WHO.

To prevent the spread of Covid-19, Governments are running several campaigns to motivate citizens to vaccinate them as soon as possible.

This paper is an attempt to study the effect of vaccination on spread of Covid-19 mathematically. It is observed that vaccine improves immunity and hence, vaccinated people get less infected even after exposure to corona virus. On flip side corona virus exposure to infection ratio is very high.

The findings shown in [3], along with the evidences for reduced viral load in vaccinated people who develop Covid-19, shows that any associated transmission risk is likely to be substantially reduced in vaccinated people.

In first section, a compartmental model is developed considering below mentioned compartments.

Susceptible population:

1) P: Unvaccinated population

2) V: Vaccinated population

Infected population:

1) E: Exposed class which contains infected but not infectious individuals

2) I: Infectious class of infected infectious individuals

Recovered class:

1) R: Individuals recovered after Covid-19.

As, Recovery after Covid-19 is not permanent. It is considered that individuals from recovered class enter the susceptible classes after recovery depending on their vaccination status.

A system of differential equations for the compartmental model is generated in this section.

In second section, points of equilibria are obtained and discussed.

In third section, criteria for feasibility of endemic equilibrium and unique DFE are discussed.

In fourth section, the basic reproduction number for a general compartmental dis- ease transmission model based on ordinary differential equation[2] is explained and Reproduction number for model developed in first section is discussed for equilibria.

In section five, stability analysis of equilibria is carried out.

2 Mathematical Modelling :

In this section a compartmental model is developed considering following compartments:

- E: Infected but not Infectious population
- P: Population without vaccination
- V: Vaccinated population
- I: Infectious population
- R: Recovered population

Here B is birth rate at which new susceptible individuals are recruiting to class P. All the compartments have mortality rate μ. Population without vaccination is getting infected at rate α and enters infected class E. Unvaccinated population gets vaccinated at rate γ. Individuals in class E are recruited to infectious class at rate δ. Infectious individuals get recovered at rate β. Recovered individuals enter vaccinated class at rate θ and leftover unvaccinated recovered individuals again enters susceptible class P.

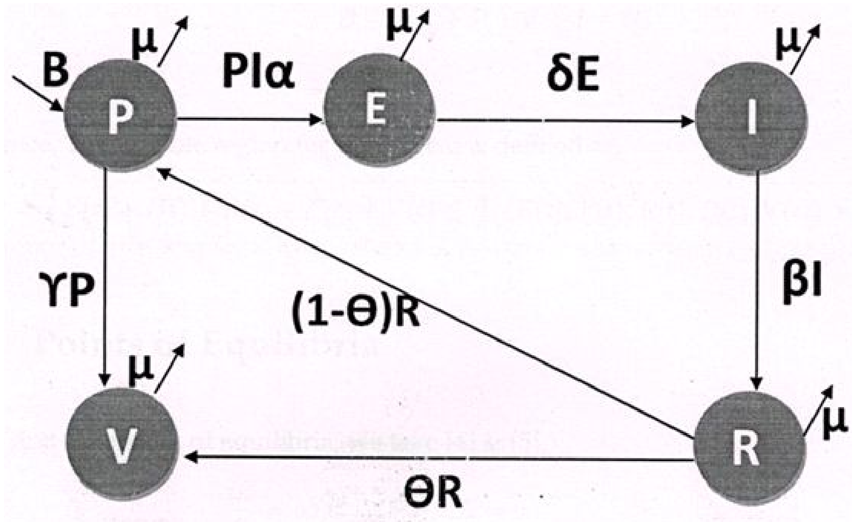

Figure-1: Compartmental Model showing movement of individuals within different classes.

From the figure we get a set of non linear differential equations shown below:

$$\frac{dE}{dt} = \alpha I P - (\delta + \mu)E$$

$$\frac{dI}{dt} = \delta E - (\beta + \mu)I$$

$$\frac{dR}{dt} = \beta I - (1 + \mu)R \quad (1)$$

$$\frac{dP}{dt} = B + (1 - \theta)R - (\alpha I + \gamma + \mu)P$$

$$\frac{dV}{dt} = \gamma P + \theta R - \mu V$$

Here, the population size at time t is,

$$N(t) = E(t) + P(t) + V(t) + I(t) + R(t)$$

Adding all the differential equations together, we get,

$$\frac{dN}{dt} = \frac{dE}{dt} + \frac{dP}{dt} + \frac{dV}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$$
$$= B - \mu(E + P + V + I + R)$$
$$= B - \mu N$$

Hence, The feasible region for the system is defined as,

$$D^* = \left\{E(t)+I(t)+R(t)+P(t)+V(t) \leq \frac{B}{\mu},\ (E(t), T(t), R(t), P(t), V(t)) > 0\right\}$$

3 Points of Equilibria :

To find the points of equilibria, we take[4 & 5],

$$\frac{dE}{dt} = \alpha IP - (\delta+\mu)E = 0$$

$$\frac{dI}{dt} = \delta E - (\beta+\mu)\mathrm{I} = 0$$

$$\frac{dR}{dt} = \beta I - (1+\mu)\mathrm{R} = 0 \qquad (2)$$

$$\frac{dP}{dt} = B + (1-\theta)\mathrm{R} - (\alpha I + \gamma + \mu)\mathrm{P} = 0$$

$$\frac{dV}{dt} = \gamma P + \theta R - \mu V = 0$$

Second equation gives rise to two cases.

1) $E = I = 0$

2) $E \neq 0$ and $I = \frac{\delta E}{\beta+\mu}$.

Case-1 *:* $E = I = 0$

Taking $E = I = 0$, we get equilibrium point

$$(E^*, I^*, R^*, P^*, V^*) = (0, 0, 0, P^*, V^*)$$

Here, we assume that, $P^* = 1$ which gives $V^* = \frac{\gamma}{\mu}$ as R^* is zero.

Hence, the point of equilibrium for case-1 is,

$$(E^*, I^*, R^*, P^*, V^*) = (0, 0, 0, 1, \frac{\gamma}{\mu})$$

which is the disease free equilibrium (DFE).

Case-2*:* $E \neq 0$ and $I = \frac{\delta E}{\beta+\mu}$

In this case by equation-1 in (2), we get,

$$P = \frac{(\delta+\mu)(\beta+\mu)}{\delta\alpha}$$

Moreover, using this expression of P and equation-4 in (2) gives,

$$R = \frac{\beta}{\alpha}\left(\frac{(\gamma+\mu)(\delta+\mu)(\beta+\mu)-\beta\delta\alpha}{\beta\delta(1-\theta)-(\delta+\mu)(\beta+\mu)(1+\mu)}\right)$$

Further, by equation-3 in (2),

$$I = \frac{1+\mu}{\beta}R$$

and

$$I = \frac{\delta E}{\beta+\mu}$$

$$\Rightarrow E = \frac{\beta+\mu}{\delta}I$$

$$= \left(\frac{\beta+\mu}{\delta}\right)\left(\frac{1+\mu}{\beta}\right)R$$

Also, by equation-5 in (2),

$$V = \frac{1}{\mu}(\gamma P + \theta R)$$

Therefore, the endemic equilibrium is obtained at

$$(E_e, I_e, R_e, P_e, V_e) = \left(\left(\frac{\beta+\mu}{\delta}\right)\left(\frac{1+\mu}{\beta}\right)R, \frac{1+\mu}{\beta}R, R, P, \frac{1}{\mu}(\gamma P + \theta R)\right)$$

where, R = $\frac{\beta}{\alpha}\left(\frac{(\gamma+\mu)(\delta+\mu)(\beta+\mu)-\beta\delta\alpha}{\beta\delta(1-\theta)-(\delta+\mu)(\beta+\mu)(1+\mu)}\right)$ and $P = \frac{(\delta+\mu)(\beta+\mu)}{\delta\alpha}$.

4 Uniqueness of DFE and Feasibility of Endemic equilibrium :

According to[5], an equilibrium point is feasible if its components are positive. Here, as α, β, δ and μ are positive, P is also positive.

As $I = \frac{1+\mu}{\beta}R$, fading of R will lead us to the DFE discussed previously. Hence R must not vanish.

Now, $R > 0$ implies,

$$(\gamma+\mu)(\delta+\mu)(\beta+\mu) - \beta\delta\alpha > 0 \text{ and } \beta\delta(1-\theta) - (\delta+\mu)(\beta+\mu)(1+\mu) > 0$$

Or

$$(\gamma+\mu)(\delta+\mu)(\beta+\mu) - \beta\delta\alpha < 0 \text{ and } \beta\delta(1-\theta) - (\delta+\mu)(\beta+\mu)(1+\mu) < 0$$

Above discussion presents proof of the theorem stated below[5].

Theorem 1. The endemic equilibrium at

$$(E_e, I_e, R_e, P_e, V_e) = \left(\left(\frac{\beta+\mu}{\delta}\right)\left(\frac{1+\mu}{\beta}\right) R, \frac{1+\mu}{\beta} R, R, P, \frac{1}{\mu}(\gamma P + \theta R) \right)$$

where, R = $\frac{\beta}{\alpha}\left(\frac{(\gamma+\mu)(\delta+\mu)(\beta+\mu)-\beta\delta\alpha}{\beta\delta(1-\theta)-(\delta+\mu)(\beta+\mu)(1+\mu)}\right)$ and $P = \frac{(\delta+\mu)(\beta+\mu)}{\delta\alpha}$ of system (1) is feasible if,

$$\alpha < \frac{(\gamma+\mu)(\delta+\mu)(\beta+\mu)}{\beta\delta} \text{ and } \theta < 1-\alpha\left(\frac{1+\mu}{\gamma+\mu}\right)$$

Or

$$\alpha > \frac{(\gamma+\mu)(\delta+\mu)(\beta+\mu)}{\beta\delta} \text{ and } \theta > 1-\alpha\left(\frac{1+\mu}{\gamma+\mu}\right)$$

5 Reproduction Number :

The basic reproduction number denoted by $\mathcal{R}_0$ is the expected number of secondary infections produced in completely susceptible population.[2] It is clear that, if $\mathcal{R}_0 < 1$ then the expected number of secondary infections is less than one. It means an infected individual infects less than one susceptible individual and hence, the disease can not grow. conversely, if $\mathcal{R}_0 > 1$, an infected individual infects more than one susceptible individual and the disease can overrun the population.

In this section, we will derive expressions to find $\mathcal{R}_0$ and $\mathcal{R}_e$ for our DFE and endemic equilibrium respectively.

To derive $\mathcal{R}_0$ and $\mathcal{R}_e$, we use two matrices shown below[2].

$$F = \begin{pmatrix} 0 & P\alpha & 0 & I\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{V} = \begin{pmatrix} \delta+\mu & 0 & 0 & 0 & 0 \\ -\delta & \beta+\mu & 0 & 0 & 0 \\ 0 & -\beta & 1+\mu & 0 & 0 \\ 0 & 0 & (\theta-1) & \mathrm{I}\alpha+\gamma+\mu & 0 \\ 0 & 0 & -\theta & -\gamma & \mu \end{pmatrix}$$

The basic reproduction number $\mathcal{R}_0$ obtained by inserting $(E^*, I^*, R^*, P^*, V^*) = (0, 0, 0, 1, \frac{\gamma}{\mu})$ in F and $\mathcal{V}$ is the spectral radius of the matrix $F\mathcal{V}^{-1}$ [2].

Here,

$$F\mathscr{V}^{-1}(E^*, I^*, R^*, P^*, V^*)$$
$$= F^*\mathscr{V}^{*-1}$$

$$= \begin{pmatrix} \frac{\alpha\delta}{(\beta+\mu)(\delta+\mu)} & \frac{\alpha(\delta+\mu+\delta\mu+\mu^2)}{(\beta+\mu)(\delta+\mu)(1+\mu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

Hence we obtain,

$$\mathscr{R}_0 = \rho(F^*\mathscr{V}^{*-1}) = \frac{\alpha\delta}{(\beta+\mu)(\delta+\mu)}$$

Now, for endemic equilibrium, the reproduction number $\mathscr{R}_e$ is the spectral radius of the matrix $F\mathscr{V}^{-1}$ obtained by considering,

$$(E_e, I_e, R_e, P_e, V_e) = \left(\left(\frac{(\beta+\mu)}{\delta}\right)\left(\frac{(1+\mu)}{\beta}\right)R, \frac{1+\mu}{\beta}R, R, P, \frac{1}{\mu}(\gamma P + \theta R)\right)$$

where, $R = \frac{\beta}{\alpha}\left(\frac{(\gamma+\mu)(\delta+\mu)(\beta+\mu)-\beta\delta\alpha}{\beta\delta(1-\theta)-(\delta+\mu)(\beta+\mu)(1+\mu)}\right)$ and $P = \frac{(\delta+\mu)(\beta+\mu)}{\delta\alpha}$ [2].

The eigen values of the matrix $F\mathscr{V}^{-1}(E_e, I_e, R_e, P_e, V_e) = F_e\mathscr{V}_e^{-1}$ are,

$$\left(0, 0, 0, 0, 1+\frac{(-B\alpha\delta+(\beta+\mu)(\gamma+\mu)(\delta+\mu)(\mu(1+\mu)(\delta+\mu)+\beta(\mu(1+\mu)+\delta(\theta+\mu)))}{\delta(\beta+\mu)(\delta+\mu)(\beta\alpha(1+\mu)+\beta(-1+\theta)(\gamma+\mu))}\right)$$

Hence,

$$Re = \rho(F_e\mathscr{V}_e^{-1})$$

$$= 1+\frac{(-B\alpha\delta+(\beta+\mu)(\gamma+\mu)(\delta+\mu)(\mu(1+\mu)(\delta+\mu)+\beta(\mu(1+\mu)+\delta(\theta+\mu)))}{\delta(\beta+\mu)(\delta+\mu)(\beta\alpha(1+\mu)+\beta(-1+\theta)(\gamma+\mu))}$$

6 Stability of Equilibria :

In this section we will discuss stability of our equilibria based on discussions carried out in[4] using Lyapunov Function.

It states that the DFE is globally asymptotically stable if $\mathscr{R}_0 < 1$ and the endemic equilibrium is globally asymptotically stable if $\mathscr{R}_e > 1$ [4].

6.1 Stability of DFE :

The expression for $\mathscr{R}_0$ derived in previous section is,

$$\mathscr{R}_0 = \frac{\alpha\delta}{(\beta+\mu)(\delta+\mu)}$$

Now, $\mathscr{R}_0 < 1$

$\Rightarrow \frac{\alpha\delta}{(\beta+\mu)(\delta+\mu)} < 1$

$\Rightarrow \alpha\delta < (\beta+\mu)\ (\delta+\mu)$

$\Rightarrow \alpha < \frac{(\beta+\mu)(\delta+\mu)}{\delta}$

This discussion gives proof to the theorem stated below[4].

Theorem 2. The DFE (0, 0, 0, 1, $\frac{\gamma}{\mu}$) of system (1) is globally asymptotically stable if,

$$\alpha < \frac{(\beta+\mu)(\delta+\mu)}{\delta}$$

where α is the rate at which unvaccinated individuals get infected.

As α is the rate of vaccination among unvaccinated individuals, if this rate is restricted to $\frac{(\beta+\mu)(\delta+\mu)}{\delta}$ then the disease free equilibrium will be stable and the disease will leave the population.

6.2 Stability of Endemic equilibrium :

In previous section we obtained that, the reproduction number for endemic equilibrium $\left(\left(\frac{(\beta+\mu)}{\delta}\right)\left(\frac{(1+\mu)}{\beta}\right)R, \frac{1+\mu}{\beta}\ R,\ R,\ P,\ \frac{1}{\mu}(\gamma P+\theta R)\right)$

where, $R = \frac{\beta}{\alpha}\ \left(\frac{(\gamma+\mu)(\delta+\mu)(\beta+\mu)-\beta\delta\alpha}{\beta\delta(1-\theta)-(\delta+\mu)(\beta+\mu)(1+\mu)}\right)$ and $P = \frac{(\delta+\mu)(\beta+\mu)}{\delta\alpha}$ is,

$$Re = 1 + \frac{(-B\alpha\delta + (\beta+\mu)(\gamma+\mu)(\delta+\mu)(\mu(1+\mu)(\delta+\mu) + \beta(\mu(1+\mu) + \delta(\theta+\mu)))}{\delta(\beta+\mu)(\delta+\mu)(\beta\alpha(1+\mu) + \beta(-1+\theta)(\gamma+\mu))}$$

Now,

$Re > 1$ implies, $\frac{(-B\alpha\delta + (\beta+\mu)(\gamma+\mu)(\delta+\mu)(\mu(1+\mu)(\delta+\mu) + \beta(\mu(1+\mu) + \delta(\theta+\mu)))}{\delta(\beta+\mu)(\delta+\mu)(\beta\alpha(1+\mu) + \beta(-1+\theta)(\gamma+\mu))} > 0$

For simplicity, we rewrite above expression as below.

$$\frac{(-B\alpha\delta + (\beta+\mu)(\gamma+\mu)(\delta+\mu)(\mu(1+\mu)(\delta+\mu) + \beta(\mu(1+\mu) + \delta(\theta+\mu)))}{\delta(\beta+\mu)(\delta+\mu)(\beta\alpha(1+\mu) + \beta(-1+\theta)(\gamma+\mu))} = \frac{a_1 a_2}{a_3}$$

where,

$a_1 = (-B\alpha\delta + (\beta+\mu)(\gamma+\mu)(\delta+\mu))$

$a_2 = (\mu(1+\mu)(\delta+\mu) + \beta(\mu(1+\mu) + \delta(\theta+\mu)))$

$a_3 = \delta(\beta+\mu)(\delta+\mu) + (\beta\alpha(1+\mu) + \beta(-1+\theta)(\gamma+\mu))$

We must note that, a_2 is always positive, as $\mu, \delta, \beta, \theta > 0$.

Hence, $\frac{a_1 a_2}{a_3} > 0$ if and only if,

$$a_1 > 0 \ \& \ a_3 > 0$$
$$\text{or}$$
$$a_1 < 0 \ \& \ a_3 < 0$$

Now, $a_1 > 0 \Rightarrow \gamma > \frac{\beta\alpha\delta}{(\beta+\mu)(\delta+\mu)} - \mu$ and $a_1 < 0 \Rightarrow \gamma < \frac{\beta\alpha\delta}{(\beta+\mu)(\delta+\mu)} - \mu$

Similarly, $a_3 > 0 \Rightarrow \delta(\beta+\mu)(\delta+\mu)(\beta\alpha(1+\mu) + \beta(-1+\theta)(\gamma+\mu)) > 0$

This implies, $a_3 > 0$ if $\theta > 1 - \frac{\alpha(1+\mu)}{\gamma+\mu}$ and $a_3 < 0$ if $\theta < 1 - \frac{\alpha(1+\mu)}{\gamma+\mu}$.

This provide proof to below theorem[4],

Theorem 3. The endemic equilibrium

$$\left(\left(\frac{(\beta+\mu)}{\delta}\right)\left(\frac{(1+\mu)}{\beta}\right)R, \frac{1+\mu}{\beta} R, R, P, \frac{1}{\mu}(\gamma P + \theta R)\right)$$

where, $R = \frac{\beta}{\alpha}\left(\frac{(\gamma+\mu)(\delta+\mu)(\beta+\mu)-\beta\delta\alpha}{\beta\delta(1-\theta)-(\delta+\mu)(\beta+\mu)(1+\mu)}\right)$ and $P = \frac{(\delta+\mu)(\beta+\mu)}{\delta\alpha}$ of system (1) is globally asymptotically stable if,

$$\gamma > \frac{\beta\alpha\delta}{(\beta+\mu)(\delta+\mu)} - \mu \ \& \ \theta > 1 - \frac{\alpha(1+\mu)}{\gamma+\mu}$$
$$OR$$
$$\gamma < \frac{\beta\alpha\delta}{(\beta+\mu)(\delta+\mu)} - \mu \ \& \ \theta < 1 - \frac{\alpha(1+\mu)}{\gamma+\mu}$$

where γ and θ indicates rates of recruitment in vaccinated compartment from unvaccinated and recovered compartments respectively.

Conclusion

In this paper stability of Disease free equilibrium and Endemic equilibrium is studied using reproduction number. Through the theorems presented in this paper, we conclude that the disease free state is stable if individuals get infected at a rate less than $\frac{(\beta+\mu)(\delta+\mu)}{\delta}$.

Where, β is rate of recovery, δ is rate at which infected individuals become infectious and μ is mortality rate.

Moreover, through Theorem - 3 it is shown that the endemic equilibrium is stable if the rates of vaccination amongst unvaccinated and recovered individuals are restricted to $\frac{\beta\alpha\delta}{(\beta+\mu)(\delta+\mu)} - \mu$ and $1 - \frac{\alpha(1+\mu)}{\gamma+\mu}$ respectively.

Scope of future work :

- This paper is an attempt to study effect of vaccination on transmission of CORONA virus. More complex models including different variants of CORONA virus and effect of vaccination on their transmission is to be studied.
- Analysis of transmission of CORONA virus at different places during different time durations can be done using results found in this paper and the actual data gathered & published by government.

References

1 . Cohen J. Normile D. New SARS-like virus in China triggers alarm, Science, (2020).
2. P. Van den Driessche, James Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental model of disease transmission, Mathematical Biosciences *180,* 29-48 (2002).
3. Science Brief : COVID-19 Vaccines and Vaccination, https://www.cdc.gov/coronavirus/2019-ncov/science/science-briefs/fully- vaccinated-people.html, 27th May 2021.
4. M. M. Ojo, F. O. Akinpelu, Lyapunov Functions and Global Properties of SEIR Epidemic Model, International journal of Chemistry, Mathematics and Physics (IJCMP), AI Publications, Vol-1(pages 1-10), May-June-2017.
5. Abdelkader Intissar, A Mathematical Study of a Generalized SEIR Model of COVID-19, Sci Medicine Journal, August 2020, 30-67.