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Weakly Connected Closed Geodetic Numbers of the Corona and Composition of Some Graphs

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Abstract

For two vertices u and v of a connected simple graph G, the closed interval $I_G[u,v]$ consists of u,v and all vertices lying in some u-v geodesic in G, while for $S \subseteq V(G)$, the set $I_G[S]$ is the union of all sets $I_G[u,v]$ for $u,v \in S$. In this paper, select vertices of G sequentially as follows: select avertex v_1 and let $S_1 = \{v_1\}$. Select a vertex $v_2 \neq v_1$ and let $S_2=\{v_1,v_2\}$, then determine $I_G[S_2]$. If $I_G[S_2] \neq V(G)$, then successively select a vertex $v_i \notin I_G[S_{i-1}]$ and let $S_i=\{v_1,v_2,...,v_i\}$ for i=3,4,...,k. Then determine $I_G[S_i]$.

A subset S of V (G) is called a weakly connected closed geodetic set of G if the selection of vertex v_k in the given manner yields $I_G[S_k]= V(G)$, where $S_k = S$, and $\langle S\rangle_w$ is connected, where $\langle S\rangle_w=\langle N[S], E_w\rangle$ with E_w consists of edges $uv \in E(G)$ such that $u \in S$ or $v \in S$.

The minimum cardinality of weakly connected closed geodetic set is called the weakly connected closed geodetic number wcgn(G) of G. In this paper, the weakly connected closed geodetic sets of the corona and composition of some graphs are characterized and the weakly connected closed geodetic numbers of these graphs are determined.

Key words: closed geodetic number of graph, weakly connected closed geodetic set, weakly connected closed geodetic number.

Introduction

The distance $d_G(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest path joining u and v in G. A u-v path of length $d_G(u,v)$ is also referred to as u-v geodesic. The neighborhood of $v \in V(G)$ is the set $N_G(v)=N(v)=\{u \in V(G): uv \in E(G)\}$. If $S \subseteq V(G)$, then the open neighborhood of S is the set $N_G(S)=N(S)=\bigcup_{v \in S} N_G(v)$. The closed neighborhood of S is $N_G[S]=N[S]=S \cup N(S)$.

A subset S of V(G) is a geodetic set of G, denoted by g-set, if $I_G[S]=V(G)$, where $I_G[S]=\bigcup\{I_G[u,v]:u,v \in S\}$. $I_G[S]$ is called the geodetic closure of S. The geodetic number of G denoted by g(G) is given by

$$g(G)=\min\{|S|:I_G[S]=V(G)\}.$$

Thus, a geodetic set S of G with $|S|=g(G)$ is called a geodetic basis of G. The set S is a closed geodetic cover of G if $S=\{v_1,v_2,...,v_k\}$ and is obtained by choosing the vertices $v_1,v_2,...,v_k$ such that the following hold:

1. $v_1 \neq v_2$;
2. $v_i \notin I_G[S_{i-1}]$ for $3 \leq i \leq k$; and
3. $I_G[S_k]=V(G)$.

The closed geodetic number of G, is given by

$$cgn(G)=\min\{|S|:S \in C^*(G)\}$$

A set $S \in C^*(G)$ with $|S|=cgn(G)$ is called the closed geodetic basis of G denoted by cgb(G). The collection of all closed geodetic covers of G denoted by $C^*(G)$.

Let $S \subseteq V(G)$. The symbol $\langle S\rangle_w=\langle N[S], E_w\rangle$ denotes the weakly induced subgraph of G with vertex set N[S] and whose edge set is

$$E_w=\{uv \in V(G):u \in S \text{ or } v \in S\}.$$

The set S is called a weakly connected closed geodetic set of G, denoted by wcg-set, if it satisfies the following properties:

1. $S \in C^*(G)$.
2. $\langle S\rangle_w$ is connected.

The minimum cardinality of a weakly connected closed geodetic set is called the weakly connected closed geodetic number of G, denoted by wcgn(G).

Let G be a connected graph and $S \subseteq V(G)$. A 2-path closure $P_2[S]_G$ of set $S \subseteq V(G)$ is the set

$$P_2[S]_G=S \cup \{w \in V(G):w \in I_G(u,v) \text{ with } d_G(u,v)=2 \text{ for some } u,v \in S\}.$$

The set S is called a 2-path closure absorbing if $P_2[S]_G=V(G)$.

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This section discusses the characteristics of weakly connected geodetic setusing the

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concept of 2-path closure absorbing set and determines the weakly connected closed geodetic number of a graph obtained from the corona of two graphs.

For simplicity and convenience, given GoH and $v \in V(G)$, we denote by H^v that copy of H whose every vertex is being joined to vertex v. We also put $v+H^v = \langle V(H^v) \cup \{v\} \rangle$.

The following lemma proved by Cagaanan, Gilbert in[2] is used to prove the next result.

Lemma 2.1[2] Let H be a nontrivial connected graph and $S \subseteq V(GoH)$. If S is a closed geodetic set of G, then $S \cap V(H^v) \neq \varnothing$ for all $v \in V(G)$.

Unlike the closed geodetic set in Lemma 2.1, the weakly connected closed geodetic set S of GoH contains vertices of both graphs G and H as discussed by the following lemma.

Lemma 2.2 Let G and H benontrivial connected graphs. If $S \subseteq V$ (GoH) is a weakly connected closed geodetic set of GoH then

$S \cap V(G) \neq \varnothing$ and $S \cap V(H^v) \neq \varnothing$, for each $v \in V(G)$.

Proof: Let G and H be nontrivial connected graphs. Suppose S is a weakly connected closed geodetic set of GoH. In view of Lemma 2.1, $S \cap V(H^v) \neq \varnothing$ for each $v \in V$ (G). To prove the other case, suppose further that $S \cap V(G) = \varnothing$.

It follows that $S \subseteq V(H)$ alone. Then for any two vertices $u,v \in V(G)$ such that $uv \in E(GoH)$, $uv \notin E_w$ since $u,v \notin S$. Thus, $\langle S \rangle_w$ of GoH is disconnected, a contradiction to our assumption. Thus, $S \cap V(G) \neq \varnothing$ □.

The following theoremtaken from[6] is used to prove the next result.

Theorem 2.3 [6] Let G be a connected noncomplete graph and $S \subseteq V(G)$. If S is a closed geodetic set of G and S is a 2-path closure absorbing set in G, then S is a weakly connected closed geodetic set of G.

The following results are some characterization of a weakly connected closed geodetic set S in a graph GoH which are essential for this study.

Theorem 2.4 Let G be a nontrivial tree and H be a nontrivial noncomplete connected graph. Then $S \subseteq V$ (GoH) is a weakly connected closed geodetic set of GoH if and only if $S = \bigcup_{v \in V(G)} S_{H^v} \cup S_G$, where S_{H^v} and S_G are weakly connected closed geodetic sets of $v+H^v$ and G, for each $v \in V(G)$, respectively.

Proof: Let G be a nontrivial tree and H be a nontrivial noncomplete connected graph. Suppose that S is a wcg-set of GoH with minimum cardinality. Then by Lemma2.2, $S \cap H^v \neq \varnothing$, for each $v \in V(G)$. We let $S \cap H^v = S_{H^v}$ for each $v \in V(G)$. Note that for every $v \in V(G)$, $v+H^v = K_1+H^v$, where H is a noncomplete graph and so $diam(v+H^v)=2$. Since $diam(v+H^v)=2$ and $S_{H^v} \subset S$, for all $y \notin S_{H^v}$, there exist two vertices $x, z \in S_{H^v}$ such that path [x,y,z] is an *x-z* geodesic. It follows that for each $v \in V(G)$, S_{H^v} is a 2-path closure absorbing set in $v+H^v$ and

$I_{v+H^v}[S_{H^v}] = v+H^v$. Thus, $S_{H^v} \in C^*(v+H^v)$. By Theorem 2.3, S_{H^v} is a weakly connected closed geodetic set of $v+H^v$ for every $v \in V(G)$. Thus, $\bigcup_{v\in V(G)} S_{H^v} \subset^{-} S$. On the other hand, by Lemma 2.2, $S \cap V(G) \neq \emptyset$. We let $S \cap V(G) = S_G$. Since G is noncomplete, it follows that $|S_G| \geq 2$. Note that every *u*-*v* geodesic in G is also a u-v geodesic in GoH. Let $w \in V(G)\backslash S_G$ then by a property of a set S, there exist $u,v \in S_G$ such that $[u,w,v]$ is a *u*-*v* geodesic in GoH with $d_{GoH}(u,v) = 2$. Since $u,v \in V(G)$ and $w \in V(G)$, $[u,w,v]$ is a *u*-*v* geodesic in G with $d_G(u,v)=2$. Since w is arbitrary, $P_2[S_G]_G = V(G)$. Thus, $I_G[S_G] = V(G)$ and so, $S_G \in C^*(G)$ is a 2-path closure absorbing set in G. By Theorem 2.3, S is a weakly connected closed geodetic set of G. Therefore, combining the two results, S is the union of two sets $\bigcup_{v\in V(G)} S_{H^v}$ and S_G, that is, $S = \bigcup_{v\in V(G)} S_{H^v} \cup S_G$.

Conversely, suppose that $S = \bigcup_{v\in V(G)} S_{H^v} \cup S_G$, where $S_{H^v} \in F_{v+H^v}$ and $S_G \in F_G$. Note that for each $v \in V(G)$, $I_{v+H^v}[S_{H^v}] = V(v+H^v)$ and $I_G[S_G] = V(G)$. Hence, $I_{GoH}[S] = V(GoH)$. Thus, $S \in C^*(GoH)$. Furthermore, for each $v \in V(G)$, $\langle S_{H^v} \rangle_w$ of $v+H^v$ is connected. Now, since $P_2[S_G]_G = V(G)$, it implies that $N[S_G] = V(G)$ and since S_G is a weakly connected closed geodetic set of G, thus $\langle S_G \rangle_w$ of G is connected. Hence, $\langle S \rangle_w$ of GoH is connected. Therefore, S is a weakly connected closed geodetic set of GoH □.

Following the theorem above is a corollary which determines the weakly connected closed geodetic number of graph GoH described in Theorem 2.4.

Corollary 2.5 Let G be a nontrivial tree and H be a nontrivial noncomplete connected graph. We have

$$wcgn(GoH) = wcgn(v+H^v)\cdot|V(G)| + wcgn(G), \quad v \in V(G).$$

Proof: Let G be a nontrivial tree and H be a nontrivial noncomplete connected graph. Let

$$\delta = wcgn(v+H^v)\cdot|V(G)| + wcgn(G), \text{ for each } v \in V(G).$$

Suppose S is a weakly connected closed geodetic set of GoH with minimum cardinality. Then $|S| = wcgn(GoH)$. By Theorem 2.4, it follows that $S = \bigcup_{v\in V(G)} S_{H^v} \cup S_G$, where $S_G \in F_G$ and $S_{H^v} \in F_{v+H^v}$ for each $v \in V(G)$. Thus,

$$\begin{aligned} wcgn(GoH) &= |S| \\ &= |S_{H^v}|\cdot|V(G)| + |S_G| \\ &\geq \delta. \end{aligned}$$

And now, suppose that for each $v \in V(G)$, S_{H^v} and S_G are weakly connected closed geodetic sets of $v+H^v$ and G with minimum cardinality, respectively. And so, for each $v \in V(G)$, $|S_{H^v}| = wcgn(v+H^v)$ and $|S_G| = wcgn(G)$. Applying Theorem 2.4, $S = \bigcup_{v\in V(G)} S_{H^v} \cup S_G$ is a weakly connected closed geodetic set of GoH. Hence,

$$\begin{aligned} wcgn(GoH) &\leq |S| \\ &= |S_{H^v}|\cdot|V(G)| + |S_G| \end{aligned}$$

$=\text{wcgn}(v+H^v)\cdot|V(G)|+\text{wcgn}(G)$
$=\delta.$

Therefore, combining the two inequalities we have wcgn(GoH)=δ □.

Lemma 2.6 Let G be a nontrivial tree. Then S is a weakly connected closed geodetic set of GoK_n if and only if S= $\bigcup_{v\in V(G)} V(K_n^v) \cup S_G$ such that S_G is a weakly connected closed geodetic set of G.

Proof: Let G be a nontrivial tree. Suppose that S is a weakly connected closed geodetic set of GoK_n. Since G is connected nontrivial and K_n is complete, each v∈V(G), $v+K_n^v$ is a complete subgraph of GoK_n. Thus, each vertex in $V(K_n^v)$, for each v∈V(G), is an extreme vertex of GoK_n. In fact, Ext(GoK_n) $=\bigcup_{v\in V(G)} V(K_n^v)$ Thus, $\bigcup_{v\in V(G)} V(K_n^v) \subset S$. Since G is a tree, following the proof of Theorem 2.4, S_G is a weakly connected closed geodetic set of G. Thus, the conclusion follows.

Conversely, suppose that S= $\bigcup_{v\in V(G)} V(K_n^v) \cup S_G$ such that S_G is a weakly connected closed geodetic set of G. Note that $V(K_n)$ is a weakly connected closed geodetic set of K_n such that $\langle S_{K_n}\rangle_w \cong K_n$. Then for each v∈V(G), $v+K_n^v$ is a complete subgraph of GoK_n. Since G is a tree and S_G is a wcg-set of G, it follows that $N[S_G]=V(G)$ and $\langle S_G\rangle_w$ is connected. Hence, $I_{GoK_n}[S]=V(GoK_n)$ and $\langle S\rangle_w$ of GoK_n is connected. Therefore, S is a weakly connected closed geodetic set of GoH □.

The following theorem is a direct consequence of Lemma 2.6.

Theorem 2.7 Let G be a nontrivial tree. Then

$$\text{wcgn}(GoK_n)=|V(K_n)|\cdot|V(G)|+\text{wcgn}(G).$$

Proof: Let G be a nontrivial tree. Suppose that S is a weakly connected closed geodetic set of GoK_n with minimum cardinality. Then |S|=wcgn(GoH). By Lemma 2.6, S= $\bigcup_{v\in V(G)} V(K_n^v) \cup S_G$, where S_G is a weakly connected closed geodetic set of G. Hence,

$$\begin{aligned}\text{wcgn}(GoK_n)&=|S|\\ &=|V(K_n)|\cdot|V(G)|+|S_G|\\ &\geq|V(K_n)|\cdot|V(G)|+\text{wcgn}(G).\end{aligned}$$

On the other hand, suppose S= $\bigcup_{v\in V(G)} V(K_n^v) \cup S_G$ such that S_G is a weakly connected closed geodetic set of G with minimum cardinality. Then $|S_G|$= wcgn(G). Again, by Lemma 2.6, S is a weakly connected closed geodetic set of GoH. Thus,

$$\begin{aligned}|V(K_n)|\cdot|V(G)|+|\text{wcgn}(G)|&=|S|\\ &\geq\text{wcgn}(GoK_n).\end{aligned}$$

Combining the two inequalities, the conclusion follows □.

We know that in a graph K_n, the set $V(K_n)$ is the wcg-set of K_n with minimum cardinality. To see if this one still holds in a graph $K_n oH$. Consider the following results.

Lemma 2.8 Let H be any nontrivial

noncomplete connected graph. Then S is a weakly connected closed geodetic set of $K_n oH$ if and only if $S=\bigcup_{v\in V(K_n)} S_{H^v}\cup\{u\}$ for any $u\in V(K_n)$, where S_{H^v} is a weakly connected closed geodetic set of $v+H^v$ for each $v\in V(K_n)$.

Proof: Suppose that S is a weakly connected closed geodetic set of $K_n oH$. Since by Lemma 2.2 $S\cap V(H^v)\neq\emptyset$ for each $v\in V(K_n)$, so we let $S\cap V(H^v)= S_{H^v}$. In view of Theorem 2.4, S_{H^v} is a wcg-set of $v+H^v$ for each $v\in V(K_n)$. Again, by Lemma 2.2 $S\cap V(K_n)\neq\emptyset$. Notice that for everyn $u,v\in V(K_n)$ such that $u\neq v$, $uv\in E(K_n)$. Now, let $S\cap V(K_n)= \{u\}$ then for every $v\in V(K_n)$ such that $v\neq u$, $uv\in E(K_n)$. Also, $uv\in E(K_n oH)$. This implies that $V(K_n)= N[u]$ and for every $v\in V(K_n)\backslash\{u\}$ there exists $u\in S\cap V(K_n)$ such that $uv\in E_w$. Now, constructing $S= \bigcup_{v\in V(K_n)} S_{H^v}\cup\{u\}$ for any $u\in V(K_n)$ implies that $I_G[S]=V(K_n oH)$ and $\langle S\rangle_w$ is connected.

Conversely, suppose that $S= \bigcup_{v\in V(K_n)} S_{H^v}\cup\{u\}$ for any $u\in V(K_n)$, where SH^v is a weakly connected closed geodetic set of $v+H^v$ for each $v\in V(K_n)$. And so, for every $v\in V(K_n)$, $I_{v+H^v}[S_{H^v}]= V(v+H^v)$ and $\langle S_{H^v}\rangle_w$ is connected. Since $S\cap V(K_n)=\{u\}$, $uv\in E_w$ for all $v\in V(K_n)\backslash\{u\}$ since $u\in S$. This implies that $\langle S\rangle_w$ of $K_n oH$ is connected. Since $I_{K_n oH}[S]=V(K_n oH)$, S is a weakly connected closed geodetic set of $K_n oH$. □

Theorem 2.9 For any noncomplete connected graph H, we have

$$wcgn(K_n oH)=wcgn(v+H^v)\cdot|V(K_n)|+1.$$

Proof: Let H be any noncomplete connected graph. Let

$$\sigma=wcgn(v+H^v)\cdot|V(K_n)|+1.$$

Suppose that $S=\bigcup_{v\in V(K_n)} S_{H^v}\cup\{u\}$ for any $u\in V(K_n)$, where S_{H^v} is a weakly connected closed geodetic set of $v+H^v$ for each $v\in V(K_n)$. Then $|S|=\sigma$. By Lemma 2.8, S is a weakly connected closed geodetic set of $K_n oH$. Thus,

$$\sigma=|S| \geq wcgn(K_n oH).$$

Suppose that S is a weakly connected closed geodetic set of $K_n oH$ with minimum cardinality. Then, $|S| = wcgn(K_n oH)$. By Lemma 2.8, $S = \bigcup_{v\in V(K_n)} S_{H^v}\cup\{u\}$ for any $u\in V(K_n)$, where SH_v is a weakly connected closed geodetic set of $v+H^v$ for each $v\in V(K_n)$. Thus,

$$wcgn(K_n oH)=|S| =|S_{H^v}|\cdot|V(K_n)|+1 \geq \sigma.$$

Therefore, $wcgn(K_n oH)=wcgn(v+H^v)\cdot|V(K_n)|+1$. □

We now consider the corona of two complete graphs. It is interesting to know the characteristics of a weakly connected closed geodetic set in these graphs.

Lemma 2.10 Let K_n and K_m be any

connected complete graphs for n,m≥2. Then S is a weakly connected closed geodetic set of $K_n o K_m$ with minimum cardinality if and only if

$$S=\bigcup_{v\in V(K_n)} V(K_n^v)\cup\{u\} \text{ where } u\in V(K_n).$$

Proof: Suppose that S is a weakly connected closed geodetic set of $K_n o K_m$ with minimum cardinality. Note that for each $v\in V(K_n)$, $v+K_m^v$ is a complete subgraph of K_n o K_m. Thus, each vertex in $V(K_m^v)$, for each $v\in V(K_n)$, is an extreme vertex of K_n o K_m. In fact, $Ext(K_n \text{ o } K_m)=\bigcup_{v\in V(K_n)} V(K_m^v)$ Note that, $Ext(K_n \text{ o } K_m)=\bigcup_{v\in V(K_n)} V(K_m^v)\subset S$. On the other hand, since K_n is a complete graph, by Lemma 2.8, $S\cap V(K_n)=\{u\}$ for any $u\in V(K_n)$. Thus, combining the two results we have

$$S=\bigcup_{v\in V(K_n)} V(K_m^v)\cup\{u\} \text{ where } u\in V(K_n).$$

Conversely, suppose that $|V(K_n)|$=n and $|V(K_m)|$=m, and suppose that
m

$$S=\bigcup_{v\in V(K_n)} V(K_m^v)\cup\{u\}=\{u,w_1,w_2,\ldots,w_{nm}\}.$$

We will show first that $S\in C^*(K_n oKm)$. Let $u = v_1$ so that S_1={u} for any $u \in V(K_n)$. Then we select another vertex $w\in V(K_m^u)$ for some $u\in V(K_n)$ to be our v_2 for S_2= {u,w}. Thus we have $I_{Kn \text{ o } Km}[u,w]=\{u,w\}$. Then for any two vertices $z,w\in S$ such that $z,w\in V(K_m^v)$ for some $v\in V(K_n)$, then $I_{Kn \text{ o } Km}[z,w]=\{z,w\}$ And if $z\in V(K_n^{u_1})$ and $w\in V(K_n^{u_2})$ for distinct $u_1,u_2 \in V(K_n)$, then $I_{Kn \text{ o } Km}[z,w]=\{z,w\}\cup\{u_1,u_2\}$. Thus, $(S/\{u\})\cap I_{Kn o Km}[z,w]=\{z,w\}$ for all $z,w\in S$. The above result implies that $v_i\notin/ I_{Kn \text{ o } Km}[S_{i-1}]$ for $i=3,4,5,\ldots,nm$, where $S_{i-1}=\{v_1,v_2,\ldots,v_{i-1}\}$.

Let $v\in V(K_n)$. Pick $z\in S\cap V(K_m^v)$ and $w\in S\backslash V(K_m^v)$. Then $v\in I_{Kn \text{ o } Km}[z,w]\subseteq I_{Kn \text{ o } Km}[S]$. Hence $V(K_n)\subseteq I_{Kn \text{ o } Km}[S]$ and so $V(K_n o K_m)= I_{Kn \text{ o } Km}[S]$. This means that $S\in C^*(K_n \text{ o } K_m)$. Since $u\in S$ such that $u\in V(K_n)$ then $uv\in E_w$ for all $v\in V(K_n)\backslash\{u\}$. Thus, $\langle S\rangle_w$ of $K_n o K_m$ is connected.

Finally, suppose that there exists a proper subset S^* of S which is also a wcg-set of $K_n o K_m$. Let $z\in S$. There are two cases to consider. First, suppose that $z\in V(K_m)$. Let $w_i,w_j\in S\backslash\{z\}$. If $w_i,w_j\in V(K_m^v)$ for some $v\in V(K_n)$, then $d_{Kn o Km}(w_i,w_j)=1$, and $z\notin I_{Kn o Km}[w_i,w_j]$. If $w_i\in V(K_m^v)$ and $w_j\in V(K_m^{v*}), v\neq v*$, then $z\in I_{Kn \text{ o } Km}[w_i,w_j]$ which happens only when $z=w_i$ or $z=w_j$, which is impossible. This means that $I_{Kn \text{ o } Km}[S\backslash\{z\}]\neq V(K_n o K_m)$. Lastly, suppose $z\in V(K_n)$. Then $z\in I_{Kn \text{ o } Km}[w_i,w_j]$ for $w_i\in V(K_m^v)$ and $w_j\in V(K_m^{v*})$, $v\neq v*$. And so, $S*\in C*(K_m o K_n)$. However, for any two vertices $v,z\in V(K_n)$, $vz\notin E_w$ which implies that $\langle S*\rangle_w$ is disconnected. Therefore, for any proper subset $S*$ of $S, S*\notin F_{Kn o Km}$, and follows the desired result. □

Theorem 2.11 For n,m≥2,

$wcgn(K_n o K_m)=|V(K_m)|\cdot|V(K_n)|+1$.

Proof: Suppose that S is a weakly

connected closed geodetic set of $K_n \circ K_m$ with minimum cardinality. Then by Lemma 2.10, S= $\bigcup_{v\in V(K_n)} V(K_m^v) \cup \{u\}$ where $u\in V(K_n)$.

Thus we have,

$$\text{wcgn }(K_n \circ K_m)=|S| = \left|\bigcup_{v\in V(K_n)} V(K_m^v)\right| + |\{u\}| = |V(Km)|\cdot|V(K_n)|+1. \square$$

3 Weakly Connected Closed Geodetic Numbers of the Composition of Some Graphs:

We will only discuss here the weakly connected geodetic set of graph $G[K_n]$ and determines its weakly connected closed geodetic number.

The following two lemmas are taken from[5]. These lemmas are necessary to prove the next results.

Lemma 3.1[5] If $[u_1,u_2,...,u_k]$, $k\geq 3$, is a u_1 - u_k geodesic in G, then for any $v_1,v_2,...,v_k \in V(K_m)$, the path $[(u_1,v_1),(u_2,v_2),...,(u_k,v_k)]$ is a (u_1,v_1)-(u_k,v_k) geodesic in $G[K_m]$.

Lemma 3.2[5] If $[(u_1,v_1), (u_2,v_2),..., (u_k,v_k)]$, $k\geq 3$, is a(u_1,v_1)-(u_k,v_k) in $G[K_m]$, then $[u_1,u_2,...,u_k]$ is a u_1-u_k geodesic in G.

Lemma 3.3 Let G be a connected graph and let $S\subseteq V(G[K_m])$. Then $S\in F_{G[K_m]}$ if and only if $S=S_G\times V(K_m)$, where S_G is a weakly connected closed geodetic set of G.

Proof: Let G be a connected graph and $S\subseteq V(G[K_m])$. Suppose that S is a weakly connected closed geodetic set of $G[K_m]$. Let$(w_1,x_1),(w_3,x_3) \in$ S such that the path $[(w_1,x_1), (w_2,x_2),(w_3,x_3)]$ is a (w_1,x_1)-(w_3,x_3) geodesic in $G[K_m]$. By Lemma 3.2, $[w_1,w_2,w_3]$ is a w_1-w_3 geodesic in G. Since $w_1 \neq w_3$ and $d_G(w_1,w_3) = 2$, it follows that $w_1,w_3 \in S_G$. Consequently, (w_1, x_1), $(w_3,x_3) \in S_G \times V(K_m)$, where x_1, $x_3 \in V(K_m)$. Thus, $S \subseteq S_G \times V(K_m)$. On the other hand, suppose that (u_1,y_1), $(u_3,y_3) \in S_G \times V(K_m)$ such that the path $[u_1,u_2, u_3]$ is u_1-u_3 geodesic in G. By Lemma 3.1, the path $[(u_1,y_1), (u_2,y_2),(u_3, y_3)]$ is a(u_1,y_1)-(u_3,y_3) geodesic in $G[K_m]$ for some $y_1,y_2,y_3\in V(K_m)$. Note that $d_G(u_1,u_3)$=2, and so $d_{G[K_m]}((u_1,y_1), (u_3,y_3))$=2. Thus, (u_1,y_1), $(u_3,y_3)\in S$ and so, $S_G \times V(K_m)\subseteq$S. Therefore, $S = S_G \times V(K_m)$.

Conversely, suppose that $S=S_G\times V(K_m)$ where $S_G\in F_G$. If $x\in S_G$, then for all $v\in V(K_m)$, $(x,v)\in$S. By Lemma 3.2, for any u-v geodesic in G where $u,v\in S_G$ is also $a(u,x)$-(v,y) geodesic in $G[K_m]$ for any $x,y\in V(K_m)$. This implies that for all $w\in V(G)\backslash S_G$, there exist $u,v\in S_G$ such that $w \in I_G[u,v]$. Same rule applies in $G[K_m]$, that is, for all $(w,x_2)\in V(G[K_m])\backslash S$, there exist (u,x_1), $(v,x_3)\in S$ such that $(w,x_2)\in I_{G[K_m]}[(u,x_1),(v,x_3)]$ for any x_1,x_2, $x_3 \in V(K_m)$. Thus, $I_{G[K_m]}[S]= V(G[K_m])$. Now, for each $u\in S_G$ and distinct vertices $x,y,z\in V(K_m)$, we have $(u,x)\notin I_{G[K_m]}[(u,y), (u,z)]$. Also, since S_G is the set of all first components of S, where $S_G\in C^*(G)$, it follows that $S\in C^*(G[K_m])$. Finally, since $\langle S_G\rangle_w$ is connected, it follows that $\langle S\rangle_w$ is also connected. Therefore, S is a weakly connected closed geodetic set of $G[K_m]$ □.

Theorem 3.4 For any connected graph G,

$$wcgn(G[K_m])=wcgn(G)\cdot m.$$

*Proof :*Let S be a weakly connected closed geodetic set of $G[K_m]$ with minimum cardinality. Then |S|=wcgn($G[K_m]$). By Lemma 3.3, $S=S_G\times V(K_m)$ where S_G is a weakly connected closed geodetic set of G. Hence we have,

$$\begin{aligned} wcgn(G[K_m]) &= |S| = |S_G \times V(K_m)| \\ &= |S_G|\cdot m \\ &\geq wcgn(G)\cdot m. \end{aligned}$$

To prove the other inequality, suppose that S_G is a weakly connected closed geodetic set of G with minimum cardinality. Then $|S_G|$ =wcgn(G). By Lemma 3.3, $S_G\times V(K_m)=S$ is a weakly connected closed geodetic set of $G[K_m]$. Hence,

$$\begin{aligned} wcgn(G)\cdot m &= |S_G \times V(K_m)| \\ &= |S| \\ &\geq wcgn(G[K_m]). \end{aligned}$$

Thus combining the two inequalities, the desired conclusion is attained □.

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