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Total Outer Independent Geodetic Number of a Graph

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Abstract

We study the new concept of total outer independent geodetic number of a graph. A geodetic set $S \subseteq V$ is said to be total outer independent geodetic set if $\langle S\rangle$ has no isolated vertex and $\langle V - S\rangle$ is an independent set. We investigate the total outer independent geodetic number of some special graphs.

Key words : Crown graph, Geodetic number, Independent geodetic set, Shadow graphs.
Mathematics Subject Classification: 10C 75, 10C 10.

1 Introduction

In this paper, we use a finite undirected simple connected graph G. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic of G and for a nonempty subset S of $V(G)$, $I[S] = \cup_{u,v\in S} I[u, v]$. A set S of vertices of G is called a geodetic set in G if $I[S] = V(G)$ and a geodetic set of minimum cardinality is the geodetic number $g(G)$. The geodetic number was introduced in[3].

A vertex v is an extreme vertex in a graph G, if the subgraph induced by its neighbors is complete. The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha_0(G)$ in G. Also $\beta_0(G)$ is the minimum number of vertices in a maximum independent set of vertex of G in[4,5]. A vertex of degree one is called a pendant vertex. Caterpillar tree is a tree of order three or more, the removal of whose end-vertices produces a path called the spine of the caterpillar.

A set of vertices of G is an independent geodetic set if S is an independent set and $I[S] = V$. The minimum cardinality of an independent geodetic set is the independent geodetic number $ig(G)$, it was

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introduced by P.Kazemi and doost Ali mojdeh[2].

A geodetic set $S \subseteq V(G)$ is a total geodetic set if the subgraph $G[S]$ induced by S has no isolated vertices. The minimum cardinality of a total geodetic set is the total geodetic number $gt(G)$ and it was introduced by Abdollahzadeh Ahangar and Vladimir Samodivkin[1]. A set of vertices S in a graph G is a nonsplit geodetic set if S is a geodetic set and the subgraph $G[V - S]$ induced by $< V(G) - S >$ is connected. The minimum cardinality of a nonsplit geodetic set, denoted by $g_{ns}(G)$, is called the nonsplit geodetic number of G was studied by Tejaswini, Venkanagouda in[7].

In this paper we define total outer independent geodetic set. For any undefined term in this paper, see[4,5].

The following theorems are used in the sequel.

Theorem 1.1.[3] *For any cycle C_n of order $n \geq 3$,*

$$g(C_n) = \begin{cases} 2 & \text{if n is even,} \\ 3 & \text{if n is odd.} \end{cases}$$

Theorem 1.2.[4] *For any cycle C_n of order $n \geq 3$,*

$$\alpha_0(C_n) = \begin{cases} \frac{n}{2} & \text{if n is even,} \\ \frac{n+1}{2} & \text{if n is odd.} \end{cases}$$

Theorem 1.3.[3] *For any graph G, $\alpha_0 + \beta_0 = n$*

Theorem 1.4.[3] *Every geodetic set of a graph contains its extreme vertices.*

2 Total outer independent geodetic number of a graph :

Definition 2.1 A geodetic set $S \subseteq V$ is said to be total outer independent geodetic set [TOIGS] if $\langle S \rangle$ has no isolated vertex and $\langle V - S \rangle$ is an independent set. The minimum cardinality of a total outer independent geodetic set and it is denoted by $g_{toi}(G)$ is called the total outer independent geodetic number of G.

Example 2.2 We depicted a graph G in Figure 1 and $S_1 = \{v_1, v_2, v_4, v_5, v_7\}$ is a g_{toi}-set so that $g_{toi}(G) = 5$.

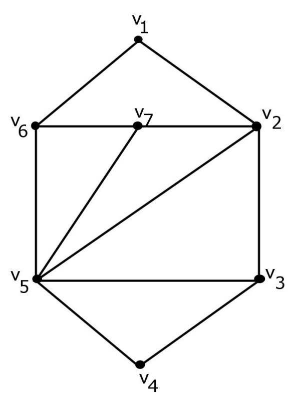

Figure 1: *G*

Remark 2.3 For the graph G given in Figure 1, $S = \{v_1, v_4, v_7\}$ is a geodetic set of G so that $g(G) = 3$. Thus the $g(G)$ and $g_{toi}(G)$ are different.

Remark 2.4 For any connected graph G, $g(G) \leq g_{toi}(G)$.

Remark 2.5 Every pendent vertex and its supporting vertex of a connected graph G belongs to every total outer independent geodetic set of G.

Remark 2.6 For any connected graph G, $2 \leq g(G) \leq ig(G) < g_{toi}(G)$.

3 Main Results

Theorem 3.1 Let $W_n = K_1 + C_{n-1}$ is the wheel, $n \geq 5$, then

$$g_{toi}(W_n) = \begin{cases} \frac{n+2}{2} & \text{if n is even,} \\ \frac{n+1}{2} & \text{if n is odd.} \end{cases}$$

Proof. Let $W_n = K_1 + C_{n-1}$ be the wheel of order $n \geq 5$ and $V(W_n) = \{x, v_1, v_2, ...v_{n-1}\}$, where $deg(x) = n-1$ and $deg(v_i) = 3$ for each $i \in \{1, 2, ...n-1\}$. We have the following cases.

Case(i) Suppose n is even. Consider $S = H_1 \cup K_1$, where H_1 is a geodetic set of W_n, with $|H_1| = \frac{n}{2}$ and $K_1 = \{x\}$ vertex. Clearly $I[S] = V[W_n]$. Hence $\langle S \rangle$ has no isolated vertex and $\langle V - S \rangle$ is an independent set. Thus S is a total outer independent geodetic set in W_n. Therefore $g_{toi}(W_n) = \frac{n}{2} + 1 = \frac{n+2}{2}$.

Case(ii) Suppose n is odd. Consider $S = H_1 \cup K_1$, where H_1 is a geodetic set of W_n, $|H_1| = \frac{n-1}{2}$ and $K_1 = \{x\}$ vertex. Clearly $I[S] = V[W_n]$. Hence $\langle S \rangle$ has no isolated vertex and $\langle V - S \rangle$ is an independent set. Thus S is a total outer independent geodetic set in W_n. Therefore $g_{toi}(W_n) = \frac{n-1}{2} + 1 = \frac{n+2}{2}$. Hence the proof.

Theorem 3.2 For any wheel $W_n = K_1 + C_{n-1}$, $n \geq 5$,

$$g_{toi}(W_n) = \begin{cases} \frac{\Delta+\delta}{2} & \text{if n is even,} \\ \frac{\Delta+\delta-1}{2} & \text{if n is odd.} \end{cases}$$

Proof. Let $W_n = K_1 + C_{n-1}$ be the wheel of order $n \geq 5$ and $V(W_n) = \{x, v_1, v_2, ...v_{n-1}\}$, where $deg(x) = n-1 > 3$ and $deg(v_i) = 3$ for each $i \in \{1, 2, ...n-1\}$. In a wheel, $\Delta = n-1$ and $\delta = 3$. We consider the following cases.

Case(i) Let n be even. We have from theorem 3.1, $g_{toi}(W_n) = \frac{n+2}{2} = \frac{(n-1)+3}{2} = \frac{\Delta+\delta}{2}$. Therefore $g_{toi}(W_n) = \frac{\Delta+\delta}{2}$.

Case(ii) Let n be odd. We have from theorem 3.1, $g_{toi}(W_n) = \frac{n+1}{2} = \frac{(n-1)+2}{2} = \frac{\Delta+\delta-1}{2}$. Therefore $g_{toi}(W_n) =$

$\frac{\Delta+\delta-1}{2}$.

Corollary 3.3: For any wheel W_n, $n \geq 5$, $g_{toi}(W_n) = g(W_n) + K_1$, where $g(W_n)$ is the geodetic number of wheel.

Theorem 3.4 : Let C_n be the cycle of order $n \geq 6$, then

$$g_{toi}(C_n) = \begin{cases} \frac{2n}{3} & \text{for } n \equiv o(\text{mod } 6) \\ \frac{2n+1}{3} & \text{for } n \equiv 1 \text{ or } 4(\text{mod } 6) \\ \frac{2n+2}{3} & \text{for } n \equiv 2 \text{ or } 5(\text{mod } 6) \\ \frac{2n+3}{3} & \text{for } n \equiv 3(\text{mod } 6) \end{cases}$$

Proof. Let V = $\{v_1, v_2, \ldots v_n\}$ be the vertex set of C_n, $n \geq 6$. We consider the following cases.

Case(i) $n \equiv 0(mod6)$. Let $S = \{v_1, v_2, v_{n/2+1}, v_{n/2+2}\}$. be the total geodetic set. But the induced subgraph $\langle V - S\rangle$ is not an independent set, which is a contradiction. Let $H = \cup v_i$, $i \neq 3k$ for all $k \in N$ and $S_1 = S \cup H$ be the total outer independent geodetic set. Thus $\langle S_1\rangle$ has no isolated vertex and $\langle V - S_1\rangle$ is an independent set. Hence $g_{toi}(C_n) = |S_1| = |S \cup H| = |S| + |H| = 4 + 2\left(\frac{n-6}{3}\right) = \frac{2n}{3}$. Therefore $g_{toi}(C_n) = \frac{2n}{3}$.

Case(ii) $n \equiv 1, 4(mod6)$. Let $S = \{v_1, v_2, v_{n+1/2}, v_{n+1/2+1},\}$ $\{v_1, v_2, v_{n/2+1}, v_{n/2+2}\}$. are the total geodetic set for $n \equiv 1(mod6)$, $n \equiv 4(mod6)$ but not a total outer independent geodetic set because induced subgraph $\langle V - S\rangle$ is not an independent set. Consider $S_1 = S \cup H$ be the total outer independent geodetic set.

Now we construct the $V(H)$ as follows?

$$H = \begin{cases} \{v_4, v_5, \ldots\ldots v_i\} & n \equiv 1(mod6), \\ \{v_4, v_5, \ldots\ldots v_i\} \cup \{v_{\frac{n}{2}} + 3, v_{\frac{n}{2}} + 4, \ldots\ldots v_j\} & n \equiv 1(mod6). \end{cases}$$

with $|H| = \frac{2n-11}{3}$, where $i \neq 3k$ for all $k \in N$ for $n \equiv 5(mod6)$ and $i \neq 3k < \frac{n}{2}$, $j \neq 3k + 1$. $\frac{n}{2} + 2 \leq j < n$ for all $k \in N$ for $n \equiv 4(mod6)$. Thus $\langle S_1\rangle$ has no isolated vertex and $\langle V - S_1\rangle$ is an independent set. Therefore $g_{toi}(C_n) = |S_1| = |S \cup H| = |S| + |H| = 4 + \frac{2n-11}{3} = \frac{2n+1}{3}$.

Case (iii) $n \equiv 2, 5(mod6)$. Let $S = \{v_1, v_2, vn/2+1, vn/2+2\} \to \{v_1, v_2, v_{n/2+1}, v_{n/2+2},\}$ $\{v_1, v_2, vn+1/2, vn+1/2+1\} \to \{v_1, v_2, v_{n+1/2}, v_{n+1/2+1}\}$ are the total geodetic set for $n \equiv 2(mod6)$, $n \equiv 5(mod6)$ but not a total outer independent geodetic set because induced sub-graph $\langle V - S\rangle$ is not an independent set. Consider $S_1 = S \cup H$ be the total outer independent geodetic set.

Now we construct the $V(H)$ as follows

$$H = \begin{cases} \{v_4, v_5, \ldots\ldots v_i\} & n \equiv 2(mod6), \\ \{v_4, v_5, \ldots\ldots v_i\} \cup \{v_{\frac{n+1}{2}} + 3, v_{\frac{n+1}{2}} + 4, \ldots\ldots v_j\} & n \equiv 4(mod6). \end{cases}$$

with $|H| = \frac{2n-10}{3}$, where $i \neq 3k$ for all $k \in N$ for $n \equiv 2(mod6)$ and $i \neq 3k < \frac{n}{2}$, $\frac{n+1}{2} < j \neq 3k + 2 < n$ for all $k \in N$ for $n \equiv 4(mod6)$. Thus $\langle S_1\rangle$ has no isolated vertex and $\langle V - S_1\rangle$ is an independent set. Therefore

$g_{toi}(C_n) = |S_1| = |S \cup H| = |S| + |H| = 4 + \frac{2n-10}{3} = \frac{2n+2}{3}$.

Case (iv) $n \equiv 3(mod6)$. Let $S = \{v_1, v_2, v_{n+1/2}, v_{n+1/2+1}\}$ be the total geodetic set. But it is not a total outer independent geodetic set. Since $\langle V - S\rangle$ is not an independent set. Now consider $S_1 = S \cup H$ be the total outer independent geodetic set. Where $H = \{v_4, v_5, ...v_i\} \cup \{v_{n+1/2+3}, v_{n+1/2+4,}....v_j\}$, $i \neq 3k < \frac{n+1}{2}$ and $\frac{n+1}{2} + 1 < j \neq 3k + 1 < n$. Thus $\langle S_1\rangle$ has no isolated vertex and $\langle V - S_1\rangle$ is an independent set. Therefore $g_{toi}(C_n) = \frac{2n+3}{3}$. Hence the proof.

The total outer independent geodetic set is not defined for P_n, $n \leq 7$. In the next theorem we discuss the result of P_n for $n \geq 8$.

Theorem 3.5 If P_n be the path of order $n \geq 8$, then

$$g_{toi}(P_n) = \begin{cases} \frac{2n}{3} + 1 & \text{for n multiple of 3,} \\ \left\lceil \frac{2n-1}{3} \right\rceil + 1 & \text{for n not multiple of 3.} \end{cases}$$

Proof. Let $P_n : v_1, v_2, ...v_n$ are the vertices of path for $n \geq 8$, we have the following cases.

Case(i) Let $n \equiv 0(mod3)$. Let $S = \{v_1, v_2, v_{n-1}, v_n\}$ be the total geodetic set, such that $\langle V - S\rangle$ is not an independent set. Consider $S_1 = S \cup A$, where $A \subseteq V(P_n) - S$, $A = \cup v_i$, $i = 3k \rightarrow i \neq 3k$ be the set it $\langle S_1\rangle$ has no isolated vertex and $\langle V(P_n) - S_1\rangle$ is an independent set. Thus $|S_1| = |S \cup A| = |S| + |A| = 4 + \frac{2n}{3} - 3 = \frac{2n}{3} + 1$. Therefore $g_{toi}(P_n) = \frac{2n}{3} + 1$.

Case(ii) Let $n \equiv 1, 2(mod3)$. Let $S = \{v_1, v_2, v_{n-1}, v_n\}$ be the total geodetic set, such that $\langle V - S\rangle$ is not an independent set, which is a contradiction. So we consider $S_1 = S \cup A$, where $A = \{v_4, v_5, ...v_i / 4 \leq i \neq 3k < n-1\}$ for all $k \in N$ with $|S_1| = \left\lceil \frac{2n-1}{3} \right\rceil + 1$. Hence the set S_1 is a total outer independent geodetic set of P_n because $\langle S_1\rangle$ has no isolated vertex and $\langle V - S_1\rangle$ is an independent set. Therefore $g_{toi}(P_n)$ = [2n-1/3+1]. Hence the proof.

Theorem 3.6 For the integers $p_1 \leq p_2 \leq, ... \leq p_k$, $p_i \geq 2$, where $1 \leq i \leq k$. Let $G = K_{p1, p2,...pk}$ be a complete k-partite graph, then $g_{toi}(G) = p_1 + p_2 + ... + p_{k-1}$.

Proof. Let $G = K_{p1, p2,...pk}$ such that $U = \{u_1, u_2,....u_{p1}\}$, $W = \{w_1, w_{2,}....w_{p2}\}$,...$X = \{x_1, x_2,...x_k\}$ where $p_1 \leq p_2 \leq, ... \leq p_k$ and $V = U \cup W \cup, ... \cup X$ is as shown in Figure 2.

Consider the geodetic set $S=U$ but $\langle S\rangle$ contains isolated vertex. Consider $S_1 = S + V(K_{p2,...pk-1})$ is a total outer independent geodetic set. Thus $\langle S_1\rangle$ has no isolated vertex and $\langle V - S_1\rangle$ is an independent set. Hence $|S_1| = |S + V(K_{p2,...pk-1})| = |S| + |V(K_{p2,...pk-1})| = p_1 + p_2 + ... + p_{k-1}$. Therefore $g_{toi}(G) = p_1 + p_2 + ... + p_{k-1}$.

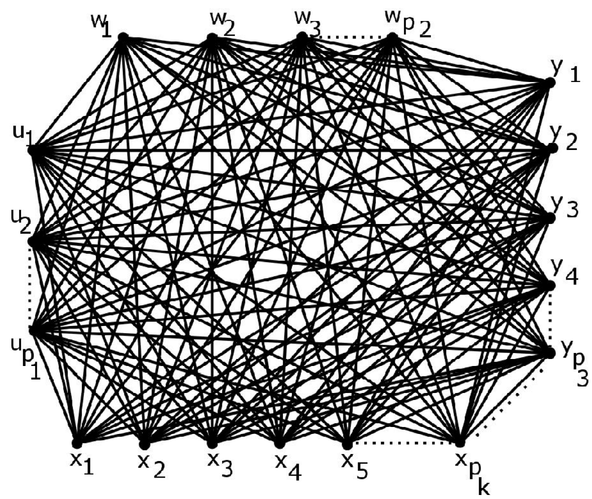

Figure 2: For Complete k-partite graph $G = K_{p1,p2,...pk}$

Theorem 3.7 If $G = K_n - \{e_i, e_j\}$ is the graph obtained from K_n by removing an non-adjacent edges $\{e_i, e_j\}$, $n \geq 5$, then $g_{toi}(G) = n - 2$.

Proof. Let $G = K_n - \{e_i, e_j\}$, where e_i, e_j are non-adjacent edges of K_n. Let $e_i = uv$ and $e_j = xy$ for some $u, v, x, y \in V(G)$, then $S = \{u, v\} = \{x, y\}$ is the geodetic set with minimum cardinality. But S has isolated vertex and $V - S$ is not an independent set. Consider $S_1 = S \cup H$, where $H \subseteq V(G) - S$ having $n - 4$ vertices and these vertices has maximum degree $n - 1$. So $\langle S_1 \rangle$ has no isolated vertex, such that $I[S_1] = V(G)$ and also the set of vertices $\langle V(G) – S_1 \rangle$ is an independent set. Thus by above argument, S_1 is a total outer independent geodetic set. Hence it follows that $|S_1| = |S \cup H| = 2+n–4 = n–2$. Therefore $g_{toi}(G) = n - 2$. Hence the proof.

Theorem 3.8 For a caterpillar tree T with all the internal vertices, v_i of $deg(v_i) \geq 3$, there is no total outer independent geodetic number.

Proof. Suppose T is a caterpillar with all the internal vertices of degree is greater than or equal to 3. Let $A = \{v_1, v_2, ...v_k\}$ be the number of k pendant vertices, $B = \{u_1, u_2, ...u_l\}$ be the number of l supporting vertices in T. Then the minimum geodetic set $S = \{v_1, v_2, ...v_k\}$, but S is not a total outer independent geodetic set, here $\langle S \rangle$ has isolated vertex and $\langle V - S \rangle$ is not an independent set. Consider $S_1 = S \cup B = V(T)$, then $V - S_1 = \phi$ does not satisfy conditions. Hence $S_1 \neq V(T)$ and $V(T) - S_1 \neq \phi$. Therefore for a caterpillar tree T with all the internal vertices, $deg(v_i) \geq 3$, there is no total outer independent geodetic number.

Theorem 3.9 If T is a caterpillar tree with atleast two internal vertices v_i, v_j such that $deg(v_i) = 2 = deg(v_j)$, $j \neq i + 1$, then $g_{toi}(T) = k + l$, where k is the number of pendant vertices and l is the number of supporting vertices.

Proof. Let T be a caterpillar tree with $A = \{v_1, v_2, ...v_k\}$, k pendent vertices and $B = \{u_1, u_2, ...u_l\}$ supporting vertices, then the geodetic set $S = \{v_1, v_2, ...v_k\}$, so $\langle S \rangle$ has isolated vertex and $\langle V(T) - S \rangle$ is not an independent set. Consider $S_1 = S \cup B$ be the total outer independent geodetic set if and only if $deg(v_i) = 2$,

$deg(v_j) = 2$ and $j \neq i+1$. Clearly $\langle S_1 \rangle$ has no isolated vertex and $\langle V(T) - S_1 \rangle$ is an independent set. Hence $g_{toi}(T) = |S_1| = |S \cup B| = |S| + |B| = k + l$. Therefore $g_{toi}(T) = k + l$.

Theorem 3.10 If T is a caterpillar tree with atleast two internal vertices v_i, v_j such that $deg(v_i) = deg(v_j) = 2$, $j = i+1$, then $g_{toi}(T) \geq k + l + 1$. where k is the number of pendant vertices and l is the number of supporting vertices.

Proof. Let T be a caterpillar tree, by theorem 3.9, it is clear that $g_{toi}(T) = k + l$ be a total outer independent geodetic number of a graph T, if and only if $deg(v_i) = deg(v_j) = 2$, $j \neq i+1$. Suppose $deg(v_i) = deg(v_j) = 2$ and $j = i+1$, then S is not a total outer independent geodetic set, since $V(T) - S$ is not an independent set. Hence $S_1 = S \cup \{v_i\}$, where v_i is the internal vertex, $deg(v_i) = 2$, be the total outer independent geodetic set of T so that $g_{toi}(T) \geq |S_1 \cup \{v_i\}| = k + l + 1$.

Theorem 3.11 For any connected graph G of order $n \geq 4$, $2 \leq g(G) \geq g_{toi}(G) \leq n \leq 2$.

Proof. A geodetic set needs atleast two vertices and therefore $g(G) \geq 2$. Since every total outer independent geodetic set is also a geodetic set of G it follows that $g(G) \leq g_{toi}(G)$. Hence $2 \leq g(G) \leq g_{toi}(G) \leq n - 2$.

Theorem 3.12 For any connected graph G of order $n \geq 4$, $3 \leq g_t(G) \leq g_{toi}(G) \leq n - 2$.

Proof. Every total geodetic set need atleast 3 vertices , so $g_t(G) \geq 3$. since every total outer independent geodetic set is a total geodetic set, we have $g_t(G) \leq g_{toi}(G)$. Since G is a connected graph, it follows that $g_{toi}(G) \leq n - 2$. Thus $3 \leq g_t(G) \leq g_{toi}(G) \leq n - 2$.

4 Shadow graph :

Definition 4.1 The shadow graph $D_2(G)$ of a graph G is obtained from G by adding, for each vertex v of G, a new vertex v', called the shadow vertex of v, and joining v' to the neighbors of v in G. Observe that (1) a vertex of G and its shadow vertex are not adjacent in $D_2(G)$ and (2) no two shadow vertices are adjacent in $D_2(G)$.

Theorem 4.2 For any path P_n, $n \geq 4$, the total outer independent geodetic number of shadow graph $D_2(P_n)$ is given by $g_{toi}(D_2(P_n)) = n + 2$.

Proof. Consider S={$v_1, v_n, v'_1 \ldots v_n'$} is the vertex set of the path P_n and $\{v'_1, v'_2, \ldots v'_n\}$ is the set of shadow vertices of path P_n respectively. Then $|V(D_2(P_n))| = 2n$ is as shown in Figure 3.

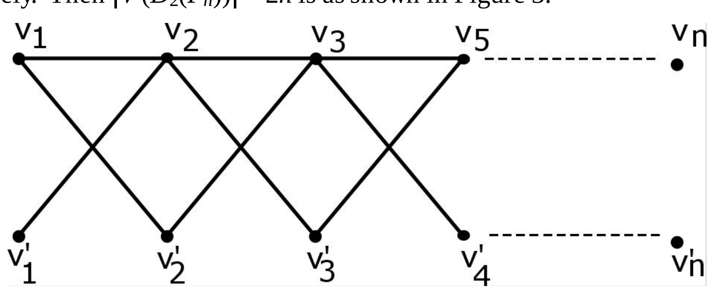

Figure 3: The shadow graph for path P_n

Let $S = \{v_1, v_n v_1', \ldots v_n'\}$ be the geodetic set, but not a total outer independent geodetic set. Since $\langle S \rangle$ has isolated vertex and $\langle V(D_2(P_n)) - S \rangle$ is not an independent set. Now consider $S' = \{v_2, v_3, \ldots v_{n-1}\} \subseteq V(D_2(P_n)) - S$ having n-2 vertices, and let $S_1 = S \cup S'$ is a total outer independent geodetic set. clearly $\langle S_1 \rangle$ has isolated vertex and $\langle V(D_2(P_n)) - S_1 \rangle$ is an independent set. Hence $|S_1| = |S \cup S'| = |S| + |S'| = 4 + n - 2 = n + 2$. Therefore $g_{toi}(D_2(P_n)) = n + 2$.

Theorem 4.3 For any cycle C_n, $n \geq 4$,

$$g_{toi}(D_2(C_n)) = \begin{cases} \Delta(D_2(C_n)) + n - 2 & if \;\; n \equiv 0 \,(mod\, 2), \\ \Delta(D_2(C_n)) + n - 1 & if \;\; n \equiv 1 (mod\, 2). \end{cases}$$

Proof. Let $G = D_2(C_n)$ be the shadow graph. Consider $\{v_1, v_n, v_1' \ldots v_n'\}$ and $\{u_1, u_2, \ldots u_n\}$ are the vertex sets of first, second copy C_n. Then $|V(D_2(C_n))| = 4n - 1$. We have two cases.

Case(i) $n \equiv 0(mod 2)$. Let $\Delta(G) = 4$ be the maximum degree of G. Let $S_1 = \{v_1, v_{n/2+1}, u_1, u_{n/2+1}\}$ be the geodetic set of G. But S_1 is not a total outer independent geodetic set. Consider $S = S_1 \cup \{v_2, \ldots v_{n/2}, v_{n/2+2}, .. v_n\}$ be the set of vertices makes $\langle S \rangle$ has no isolated vertex and $\langle V(G) - S \rangle$ is an independent set of G. Therefore $g_{toi}(D_2(C_n)) = |S| = 4 + n - 2 = \Delta(D_2(C_n)) + n - 2$.

Case (ii) $n \equiv 1(mod 2)$. Let $\Delta(G) = 4$ be the maximum degree of G. Let $S_1 = \{v_1, v_{n+1/2}, v_{n+1/2+1}, u1, u_{n+1/2}, u_{n+1/2+1},\}$ be the geodetic set of G. But $\langle S_1 \rangle$ has isolated vertex and $\langle V(G) - S_1 \rangle$ is not an independent set of G. Consider $S = S_1 \cup \{v_2, \ldots v_{\frac{n+1}{2}} - 1, v_{\frac{n+1}{2}} + 2, \ldots v_n\}$ is the set of vertices which makes a total outer independent geodetic set. Therefore $g_{toi}[D_2(C_n)] = |S| = 6 + n - 3 = \Delta(D_2(C_n)) + n - 1$.

Theorem 4.4 For star $K_{1,n}$, $n \geq 3$, $g_{toi}(D_2(K_{1,n})) = g(D_2(K_{1,n})) + 2$.

Proof. Consider two copies of $K_{1,n}$ namely $K_{1,n}$ itself and $K_{1,n}'$. Let $v_1, v_2, \ldots v_n$ are the vertices of $K_{1,n}$ with central vertex c_1 and $v_1', v_2', \ldots v_n'$ are the vertices of $K_{1,n}'$ with central vertex c_1' then $|V(D_2(K_{1,n}))| = 2n$. Consider the geodetic set $S_1 = \{\cup v_i' / 1 \leq i \leq n\}$ with minimum cardinality. But $\langle S_1 \rangle$ has isolated vertex and $< V(K_{1,n}) - S_1 >$ i is not an independent set. Consider $S = S_1 \cup \{c_1, v_i\}$ where $1 \leq i \leq n$, is the total outer independent geodetic set. Therefore $g_{toi}(D_2(K_{1,n})) = |S| = |S_1 \cup \{c_1, v_i\}| = g(D_2(K_{1,n})) + 2$.

Theorem 4.5 For star $K_{1,n}$, $n \geq 3$, $g_{toi}(D_2(K_{1,n})) = ig(D_2(K_{1,n})) + 2$.

Proof. Consider two copies of $K_{1,n}$ one is $K_{1,n}$ it self with $V(K_{1,n}) = \{v_1, v_2, \ldots v_n\}$ having central vertex c_1 and other is $K_{1,n}'$ with $V(K_{1,n}') = \{v_1', v_2', \ldots v_n'\}$ having central vertex c_1'. Let $S_1 = \{\cup v_i' / 1 \leq i \leq n\}$ be the geodetic set of $K_{1,n}$. It is obvious that $\langle S_1 \rangle$ is an independent set. So that S_1 itself forms the minimum independent geodetic set. Hence the proof follows from above theorem 4.4. That is $g_{toi}(D_2(K_{1,n})) = g(D_2(K_{1,n})) + 2 = ig(D_2(K_{1,n})) + 2$.

Corollary 4.6 For star $K_{1,n}$, $n \geq 3$, $g_{toi}(D_2(K_{1,n})) = gt(D_2(K_{1,n}))$.

5 Crown graph :

Definition 5.1 The crown graph $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

Theorem 5.2 Let $G = H_{n,n}$, be a crown graph of order $n \geq 4$ then

$$g_{toi}(H_{n,n}) = \begin{cases} 4 & \text{for n = 3,} \\ n+2 & \text{for } n \geq 3. \end{cases}$$

Proof. Let $V(G) = \{v_1, v_2, v_3, \ldots v_n, v_1', v_2', \ldots v_n'\}$ be the crown graph with $2n$ vertices is as shown in Figure 4

Case(i) For n=3 it is obvious that the set $S = \{v_1, v_n, v_1', v_n'\} \subseteq V(G)$ forms the total geodetic set and also $\langle V - S \rangle$ is an independent set. Then S itself forms the total outer independent geodetic set of G. Therefore $g_{toi}(H_{n,n}) = 4$.

Case (ii) For $n \geq 3$ let $S = \{v_i, v_i', v_j v_j'\} \subseteq V(G)$, where $1 \leq i, j \leq n$, be the minimum total geodetic set of G but the induced subgraph $\langle V - S \rangle$ is not an independent set. Consider $A = \{\cup v_k / 1 \leq k \leq n-2\} \subseteq N(v_k')$, $k = i$ or j and $B = S \cup A$ form the minimum total outer independent geodetic set of G, clearly $\langle S \rangle$ has no isolated vertex and $\langle V - B \rangle$ is an independent set. So $|B| = |S| + |A| = 4 + n - 2$. Therefore $g_{toi}(H_{n,n}) = n + 2$. Hence the proof.

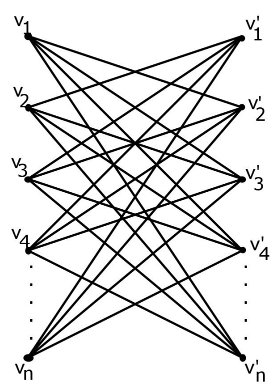

Figure 4: crown graph for $H_{n,n}$

Theorem 5.3 Let $H_{n,n}$ be a crown graph, for $n \geq 3$ then $gt(H_{n,n}) \leq g_{toi}(H_{n,n})$.

Proof. It is obvious that $S = \{u, u', v, v'\}$ is the total geodetic set and itself forms the total outer independent geodetic set of $(H_{n,n})$. Hence $gt(H_{n,n}) = g_{toi}(H_{n,n})$ holds for $n = 3$.

For $n > 3$ let $S_1 = \{v_i, v_i', v_j, v_j' / 1 \leq i, j \leq n\}$ is the minimum total geodetic set of $H_{n,n}$. Since the induced subgraph $\langle V - S_1 \rangle$ is connected, the set S_1 is not a total outer independent geodetic set of $H_{n,n}$. Now we consider a set $S_2 = S_1 \cup \{v_1, v_2, \ldots v_k\}$, where $1 \leq k \leq n-2$ which makes S_2 is a total outer independent geodetic set of $H_{n,n}$ with minimum cardinality. It is clear that $|S_1| < |S_2|$. Therefore $gt(H_{n,n}) \leq g_{toi}(H_{n,n})$. Hence the proof.

6 Conclusion

In this paper we determine the total outer independent geodetic set for crown graph, shadow graph and also we odtained the relation between total geodetic set, independent geodetic set for total outer independent geodetic set. To derive similar results in the context of other variants of geodetic number is an open area of research.

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