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The Packing Chromatic Number of Different Jump Sizes of Circulant Graphs

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Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph $G = (V,E)$ is the smallest integer k such that the vertex set $V(G)$ can be partitioned into disjoint classes $V_1,V_2,...,V_k$, where vertices in V_i have pairwise distance greater than i. In this paper, we compute the packing chromatic number of circulant graphs with different jump sizes.

Key words : Coloring; Packing chromatic number; Circulant graph.

AMS Subject Classification: 05C15, 05E30.

1. Introduction

All the graphs $G = (V, E)$ considered here are simple, finite and undirected, where $|V| = p$ denotes number of vertices and $|E| = q$ denotes number of edges of G. In general we use $\langle Vi \rangle$ to denote subgraph induced by the set of vertices V and $N(v)$ and $N[v]$ denote open and closed neighborhood of a vertex v, respectively. Let $\deg(v)$ be the degree of vertex v and usual $\delta(G)$ the minimum degree and $\Delta(G)$ the maximum degree of a graph G. A subgraph H of a graph G is called a component of G, if H is maximally connected subgraph of G. Any undefined term in this paper may be found in Harary[14].

A proper coloring of a graph G is an assignment of colors to its vertices of G, such that any two adjacent vertices have different colors. The chromatic number $\chi(G)$ is the minimum number of colors needed in a proper coloring of G. By a k-partition of G, we mean the partition $V_1,V_2,...,V_k$ of $V(G)$, where each V_i is the color class representing color i, for $i = 1,2,...,k$. Also, we say $k = \chi(G)$ or simply χ. A coloring of G using k-colors is called a k-coloring or χ-coloring of G. Similarly, the other color classes can be expected. For complete review on theory of coloring and its related parameters, we refer to[1,2,6,8,16,18].

The concept of packing chromatic number was introduced by Goddard *et al.* in[13] under the name broadcast chromatic number where an application to frequency assignments was indicated. In a given network the signals of two stations that are using the same broadcast frequency will interfere unless they are located suffciently far apart. The distance the signals will propagate is directly related to the power of those signals. Within this model all stations located at vertices in the i-packing, X_i, are allowed to broadcast at the same frequency with a power that will not allow the signals to interfere at distance i. This concept have several applications, for instance, in resource placements and biological diversity (i.e., different species in a certain area require different amounts of territory). Moreover, this concept is both a packing and a coloring (*i.e.*, a partitioning) concept. Hence, they decided to use "packing chromatic number" instead of broadcast chromatic number. The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that the vertex set $V(G)$ can be partitioned into disjoint classes $V_1,V_2,...,V_k$, where vertices in V_i have pairwise distance greater than i. The term packing chromatic number was introduced by Bresar *et al.*,[4]. For more details, we refer to[3,7,10,11,12,19. 20].

For a given positive integer p, let $s_1, s_2, \ldots, s_t$ be a sequence of integers where $0 < s_1 < s_2 < \ldots < s_t < \frac{p+1}{2}$. The circulant graph $C_p(S)$ where $S = s_1, s_2, \ldots, s_t$ is the graph on p vertices labelled as $v_1, v_2, \ldots, v_p$ with vertex v_i adjacent to each vertex $v_{i\pm s}j$ ($mod\ p$) and the values s_t are called jump sizes, see[5, 9].

The circulant graphs have a large number of applications and uses in telecommunication network VLSI design, parallel and distributed computing. The properties of circulant graphs; such as bipartitness, planarity, Hamiltonicity and colourability have been studied in[15, 17].

2. Results

2.1. Circulant graph $C_p(1)$:

The jump size of circulant graph is one, known as a cycle C_p with $p \geq 4$ vertices.

Proposition 2.1. For any circulant graph $C_p(1)$; $p \geq 4$ vertices,

$$\chi_\rho(C_p(1)) = \begin{cases} 3, if\ \ p = 4n; n \geq 1 \\ 4,\ otherwise \end{cases}$$

Proof. By [8], we have $\chi_\rho(C_p) = 3$ if $p = 4n$; $n \geq 1$ and $\chi_\rho(C_p) = 4$; otherwise. Since $C_p(1) \cong C_p$; $p \geq 4$. Hence the result follows.

2.2. Circulant graph $C_p\left(\left\lfloor \frac{p}{2} \right\rfloor\right)$

The circulant graph with jump size $\left\lfloor \frac{p}{2} \right\rfloor$; $p \geq 4$ vertices, is $C_p\left(\left\lfloor \frac{p}{2} \right\rfloor\right)$.

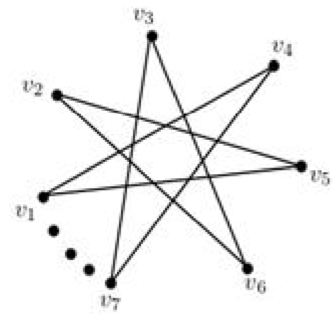

Figure 1. The circulant graph with jump size $\left\lfloor \frac{p}{2} \right\rfloor$.

Observation 2.1 :

(i). $C_p\left(\left\lfloor \frac{p}{2} \right\rfloor\right) \cong C_p(1);\ p = 2n+1,\ n \geq 1$

(ii). The circulant graph $C_p\left(\left\lfloor \frac{p}{2} \right\rfloor\right)$; $p = 2n$, $n \geq 1$ vertices contains n times of K_2' s and they are disconnected.

Proposition 2.2: For any circulant graph $C_p\left(\left\lfloor \frac{p}{2} \right\rfloor\right)$, $p \geq 4$ vertices

$$\chi_\rho\left(C_p\left(\left\lfloor \frac{p}{2} \right\rfloor\right)\right) = \begin{cases} 2, if\ \ p = 2n+2; n \geq 1 \\ 3, if\ \ p = 2n+3; n \geq 1 \end{cases}$$

Proof. Let $C_p\left(\left\lfloor \frac{p}{2} \right\rfloor\right)$ be a circulant graph with $p \geq 4$ vertices, labelled as $v_1, v_2, \ldots, v_p$. Then, we have the following casses.

Case 1. If $p = 2n + 2$; $n \geq 1$.

Since coloring the vertices $v_1, v_2, \ldots, v_p$ with the disjoint color classes $V_1, V_2, \ldots, V_k$. The jump size $\left\lfloor \frac{p}{2} \right\rfloor$ of circulant graph with $p \geq 4$ vertices contains n times of K_2's and they are disconnected. In which the sequence 1213,....starting from $v_1, v_2, \ldots, v_p$. Thus $\chi_\rho\left(C_p\left(\left\lfloor \frac{p}{2} \right\rfloor\right)\right) = 2$.

Case 2. If $p = 2n + 3$; $n \geq 1$.

In this case assign the colors from the vertices $v_1, v_2, \ldots, v_p$ with the disjoint color classes and each color class V_i is a set of vertices with the property that any distinct pair $u,v \in V_i$ satisfies $d_G(u,v) > i$. In which the sequence 1213 ... starting from v_1 in the clockwise sense. Thus $\chi_\rho\left(C_p\left(\left\lfloor \frac{p}{2} \right\rfloor\right)\right) = 3$.

2.3. Circulant graph $C_p\left(1,2,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right)$

The circulant graph with jump size 1, 2, . . . , $\left\lfloor \frac{p}{2} \right\rfloor$ **is** known as complete graph K_p with $p \geq 3$ vertices.

Proposition 2.3: For any circulant graph $C_p\left(1,2,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right)$ with $p \geq 3$ vertices,

$$\chi_\rho\left(C_p\left(1,2,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right)\right) = p.$$

Proof. By [8], we have $\chi_\rho(K_p) = p$ with $p \geq 3$ vertices. Since $C_p\left(1,2,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right) \cong K_p$. Hence the result follws.

2.4. Circulant graph with odd jump sizes :

The circulant graph with odd jump size 1,3,…, $\left\lfloor \frac{p}{2} \right\rfloor$; $p \geq 6$ vertices is known as a complete bipartite graph $K_{p1,\,p2}$ where $p_1 = p_2$; that is, $C_p\left(1,3,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right) \cong K_{p1,\,p2}$.

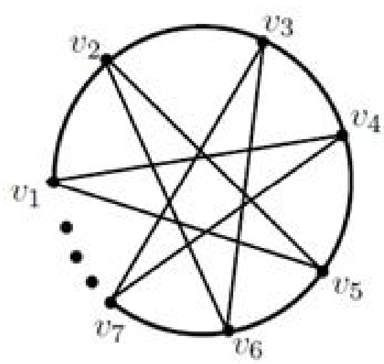

Figure 2. The circulant graph with odd jump size 1,3,…, $\left\lfloor \frac{p}{2} \right\rfloor$.

Proposition 2.4. For any circulant graph $C_p\left(1,3,\ldots,\left\lfloor \frac{p}{2}\right\rfloor\right)$; $p \geq 6$ vertices,

$$\chi_\rho\left(C_p\left(1,3,\ldots,\left\lfloor \frac{p}{2}\right\rfloor\right)\right) = \begin{cases} 3, if\ \ p = 2n+2; n \geq 2 \\ 4, if\ \ p = 2n+3; n \geq 2 \end{cases}.$$

Proof. Let $C_p\left(1,3,\ldots,\left\lfloor \frac{p}{2}\right\rfloor\right)$ be a circulant graph with $p \geq 6$ vertices, labelled as $v_1, v_2, \ldots, v_p$. Then, we have the following casses.

Case 1. If $p = 2n + 2$, $n \geq 2$

Since the circulant graph of odd jump size 1,3,…, $\left\lfloor \frac{p}{2}\right\rfloor$ with $p = 2n + 2$; $n \geq 2$ vertices and assign the colors from the even vertices. The sequence 1213,.... starting from the vertices $v_1,\ldots,v_p$. Thus $\chi_\rho\left(C_p\left(1,3,\ldots,\left\lfloor \frac{p}{2}\right\rfloor\right)\right)$=3.

Case 2. If $p = 2n + 3$, $n \geq 2$

In this case, vertex set can be partitioned in to disjoint color classes $V_1, V_2, \ldots, V_k$. The circulant graph of odd jump size 1,3,…, $\left\lfloor \frac{p}{2}\right\rfloor$ with $p = 2n + 3$; $n \geq 2$ vertices and assign the colors from the odd vertices. Each color class V_i should be an i - packing, that is, a set of vertices with the property that any distinct pair $u, v \in V_i$ satisfies dist $(u,v) > i$. Color the vertices in the sequence 1213,.... starting from the vertex v_i in the clock wise direction. Thus, $\chi_\rho\left(C_p\left(1,3,\ldots,\left\lfloor \frac{p}{2}\right\rfloor\right)\right) = 4$ follows.

2.5. Circulant graph with even jump sizes :

The circulant graph with even jump size 2,4,..., $\left\lfloor \frac{p}{2}\right\rfloor$ of circulant graph is denoted by $C_p\left(2,4,\ldots,\left\lfloor \frac{p}{2}\right\rfloor\right)$ with $p \geq 4$ vertices,

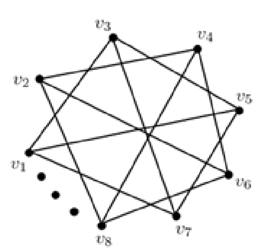

Figure 3. The circulant graph with even jump size 2,4,…, $\left\lfloor \frac{p}{2}\right\rfloor$.

Proposition 2.5. For any circulant graph $C_p\left(2,4,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right)$ with $p \geq 4$ vertices,

$$\chi_\rho\left(C_p\left(2,4,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right)\right) = \left\lceil \frac{p}{2} \right\rceil.$$

Proof. Let a circulant graph $C_p\left(2,4,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right)$; $p \geq 4$ be labelled as $v_1, v_2, \ldots, v_p$. Here the circulant graph can be partition of the vertex set of the graph into disjoint classes $V_1, V_2, \ldots, V_k$. Each color class V_i should be an i - packing, that is, a set of vertices with the property that any distinct pair $u, v \in V_i$ satisfies dist $(u,v) > i$. Let the first vertex v_1 can be colored with 1, then the sequence 1213,.... starting from v_1 in the clockwise sense. The minimum coloring set D. Hence $\chi_\rho\left(C_p\left(2,4,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right)\right) = |D| = \left\lceil \frac{p}{2} \right\rceil$.

2.6. Circulant graph $C_p(S)$ without 1

The circulant graph with jump size 2,3,4, …, $\left\lfloor \frac{p}{2} \right\rfloor$ is a $C_p\left(2,3,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right)$; $p \geq 4$ vertices.

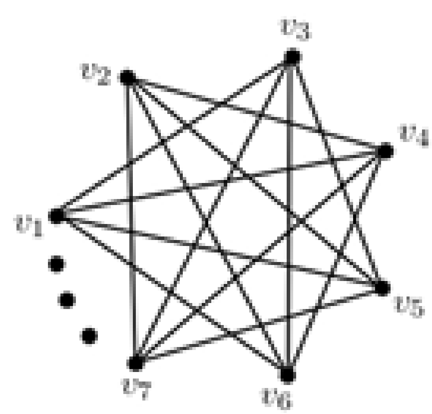

Figure 4. The circulant graph with even jump size 2,3,4,…, $\left\lfloor \frac{p}{2} \right\rfloor$.

By the similar proof techniques of Proposition 2.5, the following result is stated without proof.

Proposition 2.6. For any circulant graph $C_p\left(2,3,4,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right)$ with $p \geq 4$ vertices,

$$\chi_\rho\left(C_p\left(2,3,4\ldots,\left\lfloor\frac{p}{2}\right\rfloor\right)\right)=\left\lceil\frac{p}{2}\right\rceil.$$

2.7. Circulant graph $C_p(S)$ prime jump sizes

The circulant graph with jump size 2,3,5,…, $\left\lfloor\frac{p}{2}\right\rfloor$ is a $C_p\left(2,3,5,\ldots,\left\lfloor\frac{p}{2}\right\rfloor\right)$; $p \geq 4$ vertices.

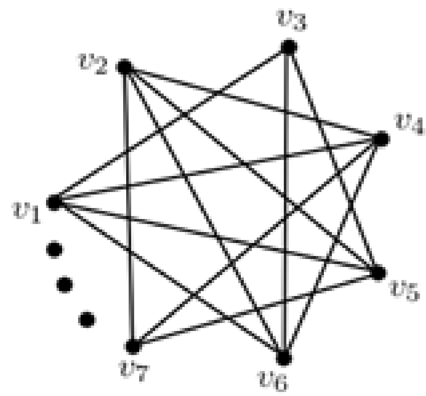

Figure 5. The circulant graph with prime jump size 2,3,5,…, $\left\lfloor\frac{p}{2}\right\rfloor$.

By the similar proof techniques of Proposition 2.5, the following result is stated without proof.

Proposition 2.7. For any circulant graph $C_p\left(2,3,5,\ldots,\left\lfloor\frac{p}{2}\right\rfloor\right)$ with $p \geq 4$ vertices,

$$\chi_\rho\left(C_p\left(2,3,5\ldots,\left\lfloor\frac{p}{2}\right\rfloor\right)\right)=\left\lceil\frac{p}{2}\right\rceil.$$

3. Conclusion and open problem

The concept of packing chromatic number (Broadcast chromatic number) have many application in resource allocations, biological diversity and some computer network. circulant graphs and their various applications are the objects of intensive study in computer science and discrete mathematics. They find the applications to the computer network design, telecommunication networking and distributed computation. Some real massively parallel processing systems, large area communication networks and projects (ILLIAC-IV, Intel Paragon, Cray T3D, MPP, SONET, MICROS, etc.) have distributed loop graphs as interconnection networks. In this paper, we enumerate the packing chromatic number of different jump sizes of circulant graphs. Further, we pose the following open problem.

Open problem 1. Obtain the packing chromatic number of the following graphs.

(i) Integral circulant graphs
(ii) Regular circulant graphs
(iii) Cayley graphs.

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References

1. M. Behzad, *Graphs and their chromatic numbers*, Doctoral thesis, Michigan State University (1965).
2. Bondy J A and Hell P, *A note on star chromatic number*, J. Graph Theory *14*, 1340-1344 (1990).
3. Bostjan Bresar, Jasmina Ferme and Sandi Klavzar, *A suvey on packing colorings*, Discussiones Mathematicae Graph theory, *40*, 923 – 970 (2020).
4. B. Bresar, S. Klavzar and D.F. Rall, *On the packing chromatic number of cartesian products, hexagonal lattice and trees*, Discrete Appl. Math. *155*, 2303-2311 (2007).
5. A. E. Brouwer, A. V. Ivanov and M. H. Klin, So*me new strongly regular graphs*, Combinatorica, *9*, 339-344 (1989).
6. B. Chaluvaraju and C. Appajigowda, *The dom-chromatic number of a graph*, Malaya J. Mat, *4(1)*, 1-7 (2016).
7. B. Chaluvaraju and M. Kumara, *The packing chromatic number of total graph of some standard graphs*, Int. J. Adv. Appl. Math. And Mech., *8(3)*, 41-47 (2021).
8. G. Chatrand and P.Zhang, *Chromatic graph theory*, CRC Press., (2006).
9. C. J. Colbourn, J. H. Dinitz, *Handbook of Combinatorial Designs*, CRC Press, (1996).
10. D.D. Durgun and H.B. Ozen Dortok, *Packing chromatic number of transformation graphs*, Thermal Sci. *23*, S1991–S1995 (2019).
11. N. Gastineau, P. Holub and O. Togni, *On the packing chromatic number of subcubic outerplanar graphs*, Discrete Appl. Math. *255*, 209 – 221 (2019).
12. N. Gastineau and O. Togni, *On S-packing edge-colorings of cubic graphs*, Discrete Appl. Math. *259*, 63-75 (2019).
13. W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, J. M. Harris and D. F. Rall, *Broadcast chromatic numbers of graphs*, Ars Combin. *86*, 33-49, (2008).
14. F. Harary, *Graph theory*, Addison-Wesley, Reading Mass, (1969).
15. P. Jalilolghadr, A. P. Kazemi, A. Khodkar, *Total dominator coloring of circulant graphs Cn(a, b)*, ar Xiv preprint, (2019).
16. T. R. Jensen and B. Toft, *Graph Coloring Problem*, John Wiley and Sons, Inc, New York (1995).
17. A. Sangeetha Devi, M. M. Shanmugapriya, Application of 2-Dominator Coloring in Graphs Using MATLAB, J. Comp. and Math. Sci, *7 (4)*, 168-174 (2016).
18. E. Sampathkumar and C. V. Venkatachalam, *Chromatic partition of a graph and graph colring and variation*, Discrete Maths., *74*(1-2), 227-239 (1989).
19. O. Tongi, *On packing colorings of distance graphs*, Discrete Appl. Math., *167*, 280 - 289 (2014).
20. A. William and S. Roy, *Packing chromatic number of certain graphs*, Internat. J. Pure Appl. Math, *87*, 731-739 (2013).