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The Distribution of the Sum of Mixed Independent Random Variables Involving Generalized H-Functions of two Variables

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Abstract

The aim of this section is to obtain the distribution of mixed sum of two independent random variables with different probability density functions. One with probability density function defined in finite range and the other with probability density function defined in infinite range and associated with product of general class of polynomials and generalized H-function of two variables. The method used is based on Laplace transform and it's inverse. The result obtained here is quite general in nature and is capable of yielding a large number of corresponding new and known results merely by specializing the parameters involved therein. To illustrate, some special cases of our main result are also given.

Keywords: H-Function, Laplace transform, random variables

1. Introduction

The generalized H-function of two variables is given by Shrivastava, H. S. P.¹⁰ and defined as follows:

$$\begin{aligned} & \mathbf{H}_{p_1, q_1; p_2, q_2; p_3, q_3}^{m_1, n_1; m_2, m_3} [x, y] \left| \begin{matrix} (a_j; \alpha_j, A_j)_{1, p_1}; (c_j, \gamma_j)_{1, p_2}; (e_j, E_j)_{1, p_3} \\ (b_j; \beta_j, B_j)_{1, q_1}; (d_j, \delta_j)_{1, q_2}; (f_j, F_j)_{1, q_3} \end{matrix} \right. \\ & = \frac{-1}{4\pi^2} L_{L_1} L_{L_2} \phi_1(\xi, \eta) \theta_2(\xi) \theta_3(\eta) x^\xi y^\eta d\xi d\eta, \end{aligned} \tag{1}$$

where

$$\phi_1(\xi, \eta) = \frac{\prod_{j=1}^{n_1} \Gamma(1 - a_j + \alpha_j \xi + A_j \eta) \prod_{j=1}^{m_1} \Gamma(b_j - \beta_j \xi - B_j \eta)}{\prod_{j=n_1+1}^{p_1} \Gamma(a_j - \alpha_j \xi - A_j \eta) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \beta_j \xi + B_j \eta)}$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_2} \Gamma(d_j - \delta_j \xi) \prod_{j=1}^{n_2} \Gamma(1 - c_j + \gamma_j \xi)}{\prod_{j=m_2+1}^{q_2} \Gamma(1 - d_j + \delta_j \xi) \prod_{j=n_2+1}^{p_2} \Gamma(c_j - \gamma_j \xi)}$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - F_j \eta) \prod_{j=1}^{n_3} \Gamma(1 - e_j + E_j \eta)}{\prod_{j=m_3+1}^{q_3} \Gamma(1 - f_j + F_j \eta) \prod_{j=n_3+1}^{p_3} \Gamma(e_j - E_j \eta)}$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non negative integers such that $p_i \geq n_i \geq 0, q_i \geq 0, q_j \geq m_j \geq 0, (i = 1, 2, 3; j = 2, 3)$. Also, all the A 's, α 's, B 's, β 's, γ 's, δ 's, E 's, and F 's are assumed to the positive quantities for standardization purpose.

The contour L_1 is in the ξ -plane and runs from $-\infty$ to $+\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \delta_j \xi)$ ($j = 1, \dots, m_2$) lie to the right, and the poles of $\Gamma(1 - c_j + \gamma_j \xi)$ ($j = 1, \dots, n_2$), $\Gamma(1 - a_j + \alpha_j \xi + A_j \eta)$ ($j = 1, \dots, n_1$) to the left of the contour.

The contour L_2 is in the η -plane and runs from $-\infty$ to $+\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j \eta)$ ($j = 1, \dots, m_3$) lie to the right, and the poles of $\Gamma(1 - e_j + E_j \eta)$ ($j = 1, \dots, n_3$), $\Gamma(1 - a_j + \alpha_j \xi + A_j \eta)$ ($j = 1, \dots, n_1$) to the left of the contour.

The generalized H-function of two variables given by (1) is convergent if

$$U = \sum_{j=1}^{m_1} \alpha_j + \sum_{j=1}^{m_1} \beta_j + \sum_{j=1}^{n_2} \gamma_j + \sum_{j=1}^{m_2} \delta_j - \sum_{j=n_1+1}^{p_1} \alpha_j - \sum_{j=m_1+1}^{q_1} \beta_j - \sum_{j=n_2+1}^{p_2} \gamma_j - \sum_{j=m_2+1}^{q_2} \delta_j; \quad (2)$$

$$V = \sum_{j=1}^{n_1} A_j + \sum_{j=1}^{m_1} B_j + \sum_{j=1}^{n_3} E_j + \sum_{j=m_1}^{m_3} F_j - \sum_{j=m_1+1}^{p_1} A_j - \sum_{j=m_1+1}^{q_1} B_j - \sum_{j=n_3+1}^{p_3} E_j - \sum_{j=m_3+1}^{q_3} F_j, \quad (3)$$

where $|\arg x| < \frac{1}{2} U\pi, |\arg y| < \frac{1}{2} V\pi$.

The distribution of sum of random variables is of great importance in many areas of physics and engineering. For example, sums of independent gamma random variables have application in problems of queuing theory such as determination of total waiting time, in civil engineering such as determination of the total excess water flow in a dam. They also appear in obtaining the inter arrival time of drought events which is the sum of the drought duration and the successive non drought duration. Various authors notably Linhart⁴, Jackson⁸ and Grice and Bain⁷ have studied the applications of distribution of sum of random variables. The distribution of the sum of two independent random variables has been obtained by several authors, particularly when both the variates come from the same family of distribution. In this context the works of Albert⁶ for uniform variates, Holm and Alouini³, and Moschopoulos⁹ for gamma variates, Loaiciga and Leipnik⁵ for Gumbel variates are worth mentioning.

Moreover, Nason² has obtained the distribution of the sum of t and Gaussian random variables and pointed out its application in Bayesian wavelet shrinkage. In recent papers Chaurasia and Kumar¹¹ and Chaurasia

and Singh¹² have investigated about the distributions of random variables associated with special functions. We know that the distribution of sum of several independent random variables when each random variable is of simply infinite or doubly infinite range can easily be obtained by means of characteristic function or moment generating function.

But, when the random variables are distributed over finite range, these methods are not much useful and the power of integral transform method comes sharply into focus. Here, we shall obtain the distribution of sum of two independent random variables, X_1 and X_2 , where X_1 possess finite uniform probability density function and X_2 follows infinite probability density function involving the product of general class of polynomials and generalized H-function of two variables, given by the equation (4) and (5) respectively.

Thus

$$f_1(x_1) = \begin{cases} \frac{1}{a}, & 0 \leq x_1 \leq a \\ 0, & \text{otherwise } a > 0 \end{cases} \tag{4}$$

$$f_2(x_2) = \begin{cases} Cx_2^{\lambda-1} e^{-\mu x_2} H_{p_1, q_1; p_2, q_2; p_3, q_3}^{m_1, n_1; m_2, n_2; m_3, n_3} [\zeta x_2^{-\gamma} \eta^{(a_j, \alpha_j; A_j)_{1, p_1}; (c_j, \gamma_j)_{1, p_2}; (e_j, E_j)_{1, p_3}}]_{(b_j, \beta_j; B_j)_{1, q_1}; (d_j, \delta_j)_{1, q_2}; (f_j, F_j)_{1, q_3}}, & x_2 \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

where

$$C^{-1} = \mu^{-\lambda} H_{p_1, q_1; p_2, q_2+1; p_3, q_3}^{m_1, n_1; m_2, n_2; m_3, n_3} [\zeta y^\gamma \eta^{(a_j, \alpha_j; A_j)_{1, p_1}; (c_j, \gamma_j)_{1, p_2}; (e_j, E_j)_{1, p_3}}]_{(b_j, \beta_j; B_j)_{1, q_1}; (d_j, \delta_j)_{1, q_2}; (\lambda, \gamma); (f_j, F_j)_{1, q_3}} \tag{6}$$

and the following conditions are satisfied:

- (i) $\gamma > 0, \mu > 0, \lambda - \gamma \min_{1 \leq j \leq m_2} (\frac{d_j}{\delta_j}) > 0,$
- (ii) $|\arg \zeta| < \frac{1}{2} U \pi, |\arg \eta| < \frac{1}{2} V \pi,$ where U and V are given in (2) and (3) respectively.
- (iii) The parameter of generalized H-function of two variables are real and so restricted that $f_2(x_2)$ remains non-negative.

2. Distribution of the Mixed Sum of two Independent Random Variables:

Theorem: If X_1 and X_2 are two independent random variables having the probability density function defined by (4) and (5) respectively. Then the probability density function of

$$Y = X_1 + X_2 \tag{7}$$

is given by $g(y) = g_1(y), 0 \leq y \leq a$
 $= g_1(y) - g_2(y), a < y < \infty$ (8)

where

$$g_1(y) = \frac{C}{a} \sum_{n=0}^{\infty} y^\lambda \frac{(-\mu y)^n}{n!} H_{p_1, q_1; p_2+1, q_2+1; p_3, q_3}^{m_1, n_1; m_2, n_2; m_3, n_3} [\zeta y^{-\gamma} \eta^{(a_j, \alpha_j; A_j)_{1, p_1}; (c_j, \gamma_j)_{1, p_2}; (1+\lambda+n, \gamma); (e_j, E_j)_{1, p_3}}]_{(b_j, \beta_j; B_j)_{1, q_1}; (d_j, \delta_j)_{1, q_2}; (\lambda+n, \gamma); (f_j, F_j)_{1, q_3}}, y \geq 0 \tag{9}$$

and

$$g_2(y) = \frac{C}{a} \sum_{n=0}^{\infty} (y - a)^\lambda \frac{\{-\mu(y-a)\}^n}{n!} H_{p_1, q_1; p_2+1, q_2+1; p_3, q_3}^{m_1, n_1; m_2, n_2; m_3, n_3} [\zeta (y-a)^{-\gamma} \eta^{(a_j, \alpha_j; A_j)_{1, p_1}; (c_j, \gamma_j)_{1, p_2}; (1+\lambda+n, \gamma); (e_j, E_j)_{1, p_3}}]_{(b_j, \beta_j; B_j)_{1, q_1}; (d_j, \delta_j)_{1, q_2}; (\lambda+n, \gamma); (f_j, F_j)_{1, q_3}}, y \geq 0 \tag{10}$$

C is given by (6) and the following conditions are satisfied:

- (i) $\gamma > 0, \mu > 0, \lambda - \gamma \min_{1 \leq j \leq m_2} \left(\frac{d_j}{\delta_j}\right) > 0,$
- (ii) $|\arg \zeta| < \frac{1}{2}U\pi, |\arg \eta| < \frac{1}{2}V\pi,$ where U and V are given in (2) and (3) respectively.
- (iii) The parameter of generalized H-function of two variables are real and so restricted that $g_1(y)$ and $g_2(y)$ remains non-negative.

Proof:

To obtain the probability density function of $Y = X_1 + X_2,$ we use the method of Laplace transform and its inverse. Let the Laplace transform of Y be denoted by $\phi_y(t),$ then

$$\phi_y(t) = L\{f_1(x_1); t\} L\{f_2(x_2); t\} \quad (11)$$

The Laplace transform of $f_1(x_1)$ is a simple integral so it can easily be evaluated and for the Laplace transform of $f_2(x_2),$ we express the generalized H-function of two variables in terms of Mellin-Barnes type contour integral (1). Now, we interchange the order of x_2 - and s-integrals and evaluate x_2 -integral as gamma integral to get

$$\phi_y(s) = \frac{C(1-e^{-at})}{a} \frac{(t+\mu)^{-\lambda}}{t} H_{p_1, q_1; p_2, q_2+1; p_3, q_3}^{m_1, n_1; m_2, n_2; m_3, n_3} \left[\begin{matrix} \zeta(t+\mu)^\gamma (a_j, \alpha_j; A_j)_{1, p_1} : (c_j, \gamma_j)_{1, p_2} : (e_j, E_j)_{1, p_3} \\ \eta (b_j, \beta_j; B_j)_{1, q_1} : (d_j, \delta_j)_{1, q_2} : (\lambda, \gamma) : (f_j, F_j)_{1, q_3} \end{matrix} \right] \quad (12)$$

Now, we break the above expression in two parts, as follows:

$$\begin{aligned} \phi_y(s) &= \frac{C}{a} \frac{(t+\mu)^{-\lambda}}{t} H_{p_1, q_1; p_2, q_2+1; p_3, q_3}^{m_1, n_1; m_2, n_2; m_3, n_3} \left[\begin{matrix} \zeta(t+\mu)^\gamma (a_j, \alpha_j; A_j)_{1, p_1} : (c_j, \gamma_j)_{1, p_2} : (e_j, E_j)_{1, p_3} \\ \eta (b_j, \beta_j; B_j)_{1, q_1} : (d_j, \delta_j)_{1, q_2} : (\lambda, \gamma) : (f_j, F_j)_{1, q_3} \end{matrix} \right] \\ &\quad - \frac{C e^{-at}}{a} \frac{(t+\mu)^{-\lambda}}{t} H_{p_1, q_1; p_2, q_2+1; p_3, q_3}^{m_1, n_1; m_2, n_2; m_3, n_3} \left[\begin{matrix} \zeta(t+\mu)^\gamma (a_j, \alpha_j; A_j)_{1, p_1} : (c_j, \gamma_j)_{1, p_2} : (e_j, E_j)_{1, p_3} \\ \eta (b_j, \beta_j; B_j)_{1, q_1} : (d_j, \delta_j)_{1, q_2} : (\lambda, \gamma) : (f_j, F_j)_{1, q_3} \end{matrix} \right] \end{aligned} \quad (13)$$

To obtain the inverse Laplace transform of first term of equation (13), we express the generalized H-function of two variables in contour integral, collect the terms involving 't' and take its inverse Laplace transform and using the known result (Erdélyi¹, p.238, eq.9). Further, writing the confluent hypergeometric function thus obtained in series form and interpreting the result by equation (1), we get the value of $g_1(y)$. The inverse Laplace transform of second term easily follows by the value of $g_1(y)$ and second shifting property for Laplace transform, we get the value of $g_2(y)$.

Conclusion

The importance of our result lies in its manifold generality. In view of the generality of the generalized H-function of two variables, on specializing the various parameters in the generalized H-function of two variables, we obtain, from our results, several pdfs such as the gamma pdf, beta pdf, Rayleigh pdf, Weibull pdf, Nakagami-m pdf, Chi-Squared pdf, half-Gaussian pdf, one-sided exponential pdf, half-Cauchy pdf, lognormal

pdf, Ricianpdf, K_pdf etc.

Thus, the results presented in this section would at once yield a very large number of pdfs occurring in the problems of statistics, applied mathematics, mathematical physics and engineering.

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