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## Technical Efficiency-Minimum Absolute Deviation and Corrected Least Squares Methods of Estimation

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### Abstract

In this paper we propose Minimum Absolute Deviation (MAD) And Corrected Least Square (CLS) Methods of Estimation to Measure Technical Efficiency of Cobb-Douglas Frontier Production Function as a Linear Programming Problem, which can be extended with comfortable ease, to any Parametric Frontier Production Function.

*Key words:* Decision Making Unit, Technical Efficiency, Cobb-Douglas Frontier production function, Minimum Absolute Deviation method, Corrected Lease Squares method.

**Mathematics Subject Classification:** 62F15, 62G07, 62-07.

### 1. Introduction

A decision making unit (DMU) such as a firm, hospital, bank, school etc., wishes to examine its performance in relation to other DMUs. The performance indicators are technical, allocative, scale and cost efficiencies. Measuring efficiency of decision making units is not only fascinating but also important for an entrepreneur or a policy maker. A decision making unit (DMU) may be a profit or non-profit oriented, a firm or a total manufacturing sector. Each DMU employs a number of inputs and produces one or more outputs. The efficiency estimation dates back to Farrell<sup>6</sup>, who in a seminal contribution proposed a method of estimation of technical, allocative and cost efficiencies. Subsequently, the scope of efficiency measurement has widened enormously, a good number of methods have emerged.

Two basic approaches to measure productive efficiency are parametric and non-parametric. The former

approach makes use of parametric frontier production functions such as Cobb-Douglas, Translog, generalized Leontiff and the Zellner-Revankar's variable return to scale frontiers and their dual cost functions. In a non-parametric approach, linear programming problems are constructed, whose constraints give rise to an empirical production and solved to assess productive efficiency of any DMU in focus.

The non-parametric approach, popularly called as Data Envelopment Analysis (DEA)<sup>5</sup> is a linear programming based method, wherein we construct linear programming problems for data on inputs and outputs and solve them to estimate pure technical and scale efficiencies following either input or output approach.

There are number of approaches to measure technical efficiency of Decision Making Units. The deterministic approach employs the linear programming technique, consequently handles empirical (non-parametric) and parametric production frontiers with comfortable ease. However, parametric estimates cannot be subjected to significance tests, due to the non-availability of standard errors.

We proposed Minimum Absolute Deviation (MAD) method of estimation of Cobb-Douglas frontier production function as a linear programming problem. However, this method can be extended with comfortable ease, to any parametric frontier production or cost function which is linear in parameters.

The Method of Corrected Least Squares can also be used for efficiency estimation, which needs to correct only the intercept but not the output elasticities to identify the appropriate production frontier. However, this method can provide an average estimate of technical efficiency.

## 2. the cobb-douglas production frontier :

Consider the Cobb-Douglas Production Frontier specification of the form

$$y = A \prod_{j=1}^n x_j^{\alpha_j}$$

For  $i^{\text{th}}$  decision making unit the Cobb-Douglas production frontier is given by,

$$\hat{y}_i = A \prod_{j=1}^n x_{ij}^{\alpha_j}$$

Taking logarithms on both sides,

$$\ln \hat{y}_i = \ln A + \sum_{j=1}^n \alpha_j X_{ij}$$

$$\hat{Y}_i = a + \sum_{j=1}^n X_{ij} \alpha_j$$

If frontier output exceeds observed output we have,

$$\hat{Y}_i \geq Y_i$$

$$a + \sum_{j=1}^n X_{ij} \alpha_j \geq Y_i \tag{2.1}$$

If there are  $k$  decision making units, then  $i=1,2,3,\dots,k$

Introducing slack variables  $s_i$ , we can express the inequation as an equation.

$$a + \sum_{j=1}^n X_{ij} \alpha_j - s_i = Y_i$$

$$\left[ a + \sum_{j=1}^n X_{ij} \alpha_j - Y_i \right] = s_i$$

A sum of these slacks give,

$$\sum_{i=1}^k s_i = ka + \sum_{i=1}^k \sum_{j=1}^n X_{ij} \alpha_j - \sum_{i=1}^k Y_i \tag{2.2}$$

The objective is minimization of sum of all slacks. By doing so we force the frontier such that all the observed outputs fall on or below the frontier.

$$\bar{s} = a + \sum_{j=1}^n \bar{X}_{.j} \alpha_j - \bar{Y} \tag{2.3}$$

Minimization of (2.2) is same as minimization of (2.3)

$\bar{Y}$  being a constant, minimization of (2.2) is same as minimization of,

$$a + \sum_{j=1}^n \bar{X}_{.j} \alpha_j \tag{2.4}$$

Combining (2.1) and (2.4) we obtain a linear programming problem for which decision variables are  $a$  and  $\alpha_j$ .

$$\begin{aligned} \text{Min} \quad & a + \sum_{j=1}^n \bar{X}_{.j} \alpha_j \\ \text{subject to} \quad & \\ & a + \sum_{j=1}^n X_{ij} \alpha_j \geq Y_i \end{aligned} \tag{2.5}$$

$$\alpha_j \geq 0, \quad a \text{ is unrestricted for sign.}$$

Letting  $a = a^+ - a^-$ , the Linear Programming Problem can be expressed as follows:

$$\begin{aligned} \text{Minimize } Z = & a^+ - a^- + \sum_{j=1}^n \bar{X}_{.j} \alpha_j \\ \text{subject to } & a^+ - a^- + \sum_{j=1}^n X_{ij} \alpha_j \geq Y_i \quad \dots(2.6) \\ & a^+, a^-, \alpha_j \geq 0 \\ & i = 1, 2, 3, \dots, k \end{aligned}$$

3. *minimum absolute deviation (mad) method of estimation of cobb-douglas production frontier:*

With two errors  $u$  and  $v$ , the former being one sided and the later two sided disturbance terms, we postulate the model as follows:

$$a^+ - a^- + \sum_{j=1}^n X_{ij} \alpha_j = Y_i + u_i + v_i \quad \text{where } 0 \leq u_i < \infty$$

$$-\infty < v_i < \infty$$

For  $i^{\text{th}}$  DMU we have

$$a^+ - a^- + \sum_{j=1}^n X_{ij} \alpha_j - Y_i - u_i = v_i$$

$$\left| a^+ - a^- + \sum_{j=1}^n X_{ij} \alpha_j - Y_i - u_i \right| = |v_i|$$

$$a^+ - a^- + \sum_{j=1}^n X_{ij} \alpha_j - Y_i - u_i = v_i^+ + v_i^-$$

$$\text{where } v_i = v_i^+ + v_i^-, \quad v_i^+ = \text{Max}\{0, v_i\}, \quad v_i^- = -\text{Min}\{0, v_i\}$$

The optimization problem equivalent to MAD estimation based model, is as follows,

$$\text{Min } \sum_{i=1}^k (v_i^+ + v_i^-)$$

subject to

$$a^+ - a^- + \sum_{j=1}^n X_{ij} \alpha_j - u_i - v_i^+ - v_i^- = Y_i \quad (3.1)$$

$$a^+, a^-, \alpha_j, u_i, v_i^+, \text{ and } v_i^- \geq 0$$

$$j=1,2,3,\dots,n$$

$$i=1,2,3,\dots,m$$

- The decision variables of the Linear Programming (3.1) are ,

$$A, \alpha_j, u_i, v_i^+ \text{ and } v_i^-$$

- If  $v = v^+ - v^-$ , then it can be seen that,  $|v| = v^+ + v^-$
- The optimal solution of Linear Programming problem (3.1) reveals DMU specific technical efficiency.

$$a + \sum_{j=1}^n X_{ij} \alpha_j = Y_i + u_i + v_i$$

$$A \prod_{j=1}^n x_{ij}^{\alpha_j} = y_i \exp(u_i + v_i)$$

$$\frac{A \prod_{j=1}^n x_{ij}^{\alpha_j}}{y_i} = \exp(u_i + v_i)$$

$$\frac{A \prod_{j=1}^n x_{ij}^{\alpha_j}}{y_i} \exp(-v_i) = \exp(u_i)$$

$$\frac{A \prod_{j=1}^n x_{ij}^{\alpha_j}}{y_i} \exp(v_i^- - v_i^+) = \exp(u_i)$$

- $u_i \geq 0, \forall i \Rightarrow \exp(u_i) \geq 1$
- The parameters  $A, \alpha_j, u_i, v_i^+$  and  $v_i^-$  can all be obtained solving Linear Programming problem (3.1)

#### 4. Efficiency estimation –the role of corrected least squares:

Consider the Cobb-Douglas production function,

$$y = A \prod_{i=1}^m x_i^{\alpha_i} u \quad \text{where } 0 \leq u \leq 1 \quad (4.1)$$

Afriat(1972)<sup>2</sup> pointed out that, technical efficiency can be measured by treating  $u$  as a continuous random variable taking values of the interval  $[0,1]$

Define  $u = e^{-z}$ ;  $0 < z < \infty$

Let the random variable  $Z$  follow Gamma distribution, so that,

$$f(z, n) = \frac{1}{\Gamma(n)} z^{n-1} \exp(-z)$$

$$\text{where } \Gamma(n) = \int_0^{\infty} z^{n-1} e^{-z} dz$$

$$\ln u = -z, \quad -\ln u = z, \quad dz = -\frac{du}{u}, \quad z = \ln\left(\frac{1}{u}\right)$$

$$z = 0 \Rightarrow u = 1$$

$$z = \infty \Rightarrow u = 0$$

$$\begin{aligned} g(u, n) du &= \frac{1}{\Gamma(n)} \left( \ln \frac{1}{u} \right)^{n-1} u \frac{du}{u} \\ &= \frac{1}{\Gamma(n)} \left( \ln \frac{1}{u} \right)^{n-1} du \end{aligned}$$

The probability density function of  $u$  is given by,

$$g(u, n) = \frac{1}{\Gamma(n)} \left( \ln \frac{1}{u} \right)^{n-1} \quad (4.2)$$

- $n$  is shape parameter of the distribution,  $g(u, n)$
- $n < 1$  implies that a greater proportion of DMUs are efficient
- $n = 1$  implies uniform efficiency
- $n > 1$  implies that a greater proportion of DMUs are inefficient

The average level of efficiency of the industry comprised of several decision making units (firms) is,

$$\bar{u} = E(u) = \int_0^{\infty} \exp(-z) \frac{1}{\Gamma(n)} z^{n-1} \exp(-z) dz$$

$$= \frac{1}{\Gamma(n)} \int_0^{\infty} \exp(-2z) z^{n-1} dz$$

$$\text{put } 2z = v, \quad 2dz = dv, \quad dz = \frac{1}{2} dv$$

$$\bar{u} = \frac{1}{\Gamma(n)} \int_0^{\infty} \exp(-v) \left( \frac{v}{2} \right)^{n-1} 2^{-1} dv$$

$$= \frac{2^{-n}}{\Gamma(n)} \int_0^{\infty} \exp(-v) (v)^{n-1} dv$$

$$= \frac{2^{-n}}{\Gamma(n)} \Gamma(n)$$

$$\bar{u} = 2^{-n}$$

## 5. The method of corrected least squares

Consider the Cobb-Douglas production function specification

$$y_i = A \prod_{j=1}^m x_{ij}^{\beta_j} u_i, \quad i=1,2,3,\dots,k$$

$$\ln y_i = \ln A + \sum_{j=1}^m \beta_j \ln x_{ij} + \ln u_i$$

$$Y_i = a + \sum_{j=1}^m \beta_j X_{ij} - z_i \quad (5.1)$$

We have,

$$\begin{aligned} E(z_i) &= \frac{1}{\Gamma(n)} \int_0^{\infty} z_i z_i^{n-1} \exp(-z_i) dz_i \\ &= \frac{1}{\Gamma(n)} \int_0^{\infty} z_i^{n+1-1} \exp(-z_i) dz_i \\ &= \frac{1}{\Gamma(n)} \Gamma(n+1) \\ &= \frac{n\Gamma(n)}{\Gamma(n)} \end{aligned}$$

$$E(z_i) = n, \quad \forall n$$

$$E(z_i^2) = \frac{1}{\Gamma(n)} \int_0^{\infty} z_i^{n+2-1} \exp(-z_i) dz_i$$

$$\begin{aligned} &= \frac{\Gamma(n+2)}{\Gamma(n)} = \frac{(n+1)n\Gamma(n)}{\Gamma(n)} \\ &= n(n+1) \end{aligned}$$

$$V(z_i) = E(z_i^2) - [E(z_i)]^2 = n(n+1) - n^2$$

$$V(z_i) = n, \quad \forall i$$

We shall assume that  $\text{Cov}(z_j, z_l) = 0 \quad j \neq l$

$$\text{Define } \alpha_0 = a - n, \quad v_i = n - z_i \quad (5.2)$$

Combine (5.1) and (5.2) to obtain,

$$\hat{y}_i = \alpha_0 + \sum_{j=1}^m \alpha_j X_{ij} + v_i$$

where  $Y_i = \ln y_i$

$$X_{ij} = \ln x_{ij}$$

Let  $v_i$  be an error term that satisfies the following properties:

- $E(v_i) = 0, \quad \forall i$
- $V(v_i) = E(v_i^2) = n$
- $\text{Cov}(v_i, v_j) = 0, \quad i \neq j$
- $E\left(\frac{v_i}{X}\right) = 0$

Under these conditions the ordinary least squares estimators are best linear unbiased estimators of  $\alpha_0, \alpha_1, \dots, \alpha_m$ . Since  $n$  is variance of  $v_i$ , the OLS estimator of  $n$  is,

$$\hat{n} = \frac{\sum_{i=1}^k \left( Y_i - \hat{\alpha}_0 - \sum_{j=1}^m \hat{\alpha}_j X_{ij} \right)^2}{k - m - 1} \quad (5.3)$$

$$E(\hat{\alpha}_0) = \alpha_0 = a - n \quad (5.4)$$

$$E(\hat{\alpha}_i) = \alpha_i, i = 1, 2, 3, \dots$$

$$E(\hat{n}) = n$$

$$E(\hat{\alpha}_0) = a - E(\hat{n})$$

$$E(\hat{\alpha}_0 + \hat{n}) = a \quad (5.5)$$

$\hat{\alpha}_0 + \hat{n}$  is an unbiased estimator of  $a$

It can be shown that  $\exp(\hat{\alpha}_0 + \hat{n})$  is consistent, but upward biased estimate of  $a$ .

In a similar way,  $2^{-\hat{n}}$  turns out to be a consistent but upward biased estimator of average technical efficiency.

$$\hat{u} = 2^{-\hat{n}} \quad (5.6)$$

From knowledge of the Gamma distribution, we can estimate the proportion of DMUs with efficiency level at least equal to  $a$

$$\begin{aligned} P[u \geq d] &= P[e^{-z} \geq d] \\ &= P[z \leq -\ln d] \end{aligned} \quad (5.7)$$



$$= \int_0^{-\ln d} \frac{1}{\Gamma(n)} z^{n-1} \exp(-z) dz \quad (5.8)$$

The right hand side expression is an incomplete Gamma integral, whose value can be found using numerical methods for each choice of d.

### Conclusion

These methods can be extended with comfortable ease to any parametric frontier production functions by suitable transformations.

#### Scope of future work :

This kind of research study can be further extended by proposing some advanced stochastic cost and production frontiers using CES and Translog functional forms

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### References

1. Aigner, D.J., Lovell, C.A.K. and Schmidt, P., 'Formulation and Estimation of Stochastic Frontier Production Function Models', *Journal of Econometrics*, 6, 21-37 (1977).
2. Afriat, S., 'Efficiency Estimation of Production Functions', *International Economic Review*, 13, 568-98 (1972).
3. Banker, cooper, Seiford, Thrall and Chu, 'Returns to scale in different DEA models', *European Journal of Operations Research*, 154, 345-362 (2004).
4. Charnes, A., Cooper, W.W and Rhodes, E., 'Measuring the Efficiency of Decision Making Units', *European Journal of Operations Research*, 2, 429-444 (1978).
5. Cooper, Seiford and Tone, 'Data Envelopment Analysis', A comprehensive text with Models', *Kluwer Academic Publishers*, Boston (2003).
6. Farrell, M.J., 'The Measurement of Productive Efficiency', *Journal of Royal Statistical society, Series-A*, 120, 253-281 (1957).
7. Richmond, J., 'Estimating the Efficiency of Production', *International Economic Review*, 15, 515-521 (1974).
8. Schmidt, P., Tsai-Fen Lin, 'Simple Tests of Alternative Specifications in Stochastic Frontier models', *Journal of Econometrics*, 24, 349-61 (1984).
9. Schmidt, P., 'Frontier Production Functions', *Econometric Reviews*, 4, 289-328 (1986).
10. Zieba, M., An analysis of technical efficiency and efficiency factors for Austrian and Swiss non-profit theatres. *Swiss Journal of Economics and Statistics*, 147(II), 233-274 (2011).
11. Alexandra Maria Rios Cabral, Francisco S. Ramos, Efficiency Container ports in Brazil, a DEA and FDH approach, *The Central European Review of Economics and Management*, Vol. 2, No.1 (2018).