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Study of Kaprekar Numbers and Constants

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Abstract

In this article, Kaprekar numbers and constants in recreational number theory are discussed. By extending the definition of Kaprekar numbers to binary numbers, the Kaprekar numbers can be expressed in binary form.

Key words: Kaprekar number, Kaprekar routine, Kaprekar constant, Binary numbers.

1. Introduction

The Kaprekar numbers were introduced by Dattathreya Ramachandra Kaprekar(1905-1986). D.R.kaprekar was an Indian recreational mathematician. He was born in Dahanu, a town about 100 Km north of Mumbai, on 17 January 1905. He worked as a school teacher in Devlali, a town very close to Nasik in Maharashtra, from 1930 -1962.

A Kaprekar number for a given base is a non-negative integer, whose square can be split up into two parts, which add up to the original number again.

Let X be a non-negative integer and n is a positive integer. X is known as n- Kaprekar number for the base b, if there exist a non-negative integer A and positive integer B satisfying the following pair of equations.

$$X^2 = Ab^n + B, \text{ where } 0 < B < b^n, \; A \geq 0 \; \& \; X \geq 1 \tag{1}$$

$$X = A + B \tag{2}$$

The second part B in RHS of (2) must be non-zero.

$X = 1$ is taken as a Kaprekar number for all $n \geq 1$, because $1^2 = 0 \times b^n + 1$; $1 = 0 + 1$. Here $A = 0$ and $B = 1$.

$X = 4879$ is a 5 - Kaprekar number for base $b = 10$.

$4879^2 = 23804641 = 238 \times 10^5 + 4641$; $238 + 4641 = 4879$. In this case $A = 238$ and $B = 4641$.

$X =$ 181819 is a 6 - Kaprekar number for base $b = 10$.

$181819^2 = 33058148761 = 33058 \times 10^6 + 148761$; $33058 + 148761 = 181819$. Here $A = 33058$ and $B = 148761$.

The rest of the article is organized as follows. The sequence of Kaprekar numbers is given in Section 2. Generation of Kaprekar numbers is mentioned in Section 3. Some properties of Kaprekar numbers are given in Section 4. Kaprekar routine and Kaprekar constants are given in Section 5. The Section 6 presents Kaprekar numbers with binary base 2. Conclusion is given in Section 7. Finally references are given in Section 8.

2.The sequence of Kaprekar Numbers :

The sequence of Kaprekar numbers in base 10 is

1, 9, 45, 55, 99, 297, 703, 999, 2223, 2728, 4879, 4950, 5050, 5292, 7272, 7777, 9999, 17344, 22222, 77778, 82656, 95121, 99999, 142857, 148149, 181819, 187110, 208495, 318682, 329967, 351352, 356643, 390313, 461539, 466830, 499500, 500500,533170, 538461, 609687, 643357, 648648, 670033, 681318, 791505, 812890, 818181, 851851, 857143, 961038, 994708, 999999, 4444444, 4927941, 5072059, 5555556, 9372385, 9999999, 11111112, 13641364, 16590564, 19273023, 19773073, 24752475, 25252525, 30884184, 36363636, 38883889, 44363341, 44525548, 49995000, 50005000, 55474452, 55636659, 61116111, 63636364, 69115816, 74747475, 75247525, 80226927, 80726977, 83409436, 86358636, 88888888, 91838088, 94520547, 99999999, 234567901, 332999667, 432432432, 567567568, 667000333, 765432099, 999999999, 1111111111, 1776299581, 2020202020, 3846956652, 3888938889, 4090859091, 4132841328,..........

The following expansions show that the numbers in this series are Kaprekar numbers as per (1) and (2) for base 10.

$1^2 = 1 = 0 \times 10^2 + 1$; $0 + 1 = 01$

$9^2 = 81 = 8 \times 10^2 + 1$; $8 + 1 = 9$

$45^2 = 2025 = 20 \times 10^2 + 25$; $20 + 25 = 45$

$55^2 = 3025 = 30 \times 10^2 + 25$; $30 + 25 = 55$

$99^2 = 9801 = 98 \times 10^2 + 01$; $98 + 01 = 99$

$297^2 = 88209$; $88 + 209 = 297$

$703^2 = 494209$; $494 + 209 = 703$

$999^2 = 998001$; $998 + 001 = 999$

$2223^2 = 4941729$; $494 + 1729 = 2223$

$2728^2 = 7441984$; $744 + 1984 = 2728$

$4879^2 = 23804641$; $238 + 04641 = 4879$

$4950^2 = 24502500$; $2450 + 2500 = 4950$

$5050^2 = 25502500$; $2550 + 2500 = 5050$

$5292^2 = 28005264$; $28 + 005264 = 5292$

$7272^2 = 52881984$; $5288 + 1984 = 7272$

$7777^2 = 60481729$; $6048 + 1729 = 7777$

$9999^2 = 99980001$; $9998 + 0001 = 9999$

$17344^2 = 300814336$; $3008 + 14336 = 17344$

$22222^2 = 493817284$; $4938 + 17284 = 22222$

$77778^2 = 6049417284$; $60494 + 17284 = 77778$

$82656^2 = 6832014336$; $68320 + 14336 = 82656$
$95121^2 = 9048004641$; $90480 + 04641 = 95121$
$99999^2 = 9999800001$;$99998 + 00001 = 99999$
$142857^2 = 20408122449$; $20408 + 122449 = 142857$
$148149^2 = 21948126201$; $21948 + 126201 = 148149$
$181819^2 = 33058148761$; $33058 + 148761 = 181819$
$187110^2 = 35010152100$; $35010 + 152100 = 187110$
$208495^2 = 43470165025$; $43470 + 165025 = 208495$
$318682^2 = 101558217124$; $101558 + 217124 = 318682$
$329967^2 = 108878221089$; $108878 + 221089 = 329967$
$351352^2 = 123448227904$; $123448 + 227904 = 351352$
$356643^2 = 127194229449$; $127194 + 229449 = 356643$
$390313^2 = 152344237969$; $152344 + 237969 = 390313$
$461539^2 = 213018248521$; $213018 + 248521 = 461539$
$466830^2 = 217930248900$; $217930 + 248900 = 466830$
$499500^2 = 249500250000$;$249500 + 250000 = 499500$
$500500^2 = 250500250000$; $250500 + 250000 = 500500$
$533170^2 = 284270248900$; $284270 + 248900 = 533170$
$538461^2 = 289940248521$; $289940 + 248521= 538461$
$609687^2 = 371718237969$; $371718 + 237969 = 609687$
$643357^2 = 413908229449$; $413908 + 229449 = 643357$
$648648^2 = 420744227904$; $420744 + 227904 = 648648$
$670033^2 = 448944221089$; $448944 + 221089 = 670033$
$681318^2 = 464194217124$; $464194 + 217124 = 681318$
$791505^2 = 626480165025$; $626480 + 165025 = 791505$
$812890^2 = 660790152100$; $660790 + 152100 = 812890$
$818181^2 = 669420148761$; $669420 + 148761 = 818181$
$851851^2 = 725650126201$; $725650 + 126201 = 851851$
$857143^2=734694122449$; $734694 + 122449 = 857143$
$961038^2=923594037444$; $923594 + 037444 = 961038$
$994708^2=989444005264$; $989444 + 005264 = 994708$
$999999^2=999998000001$; $999998 + 000001 = 999999$
$4444444^2=19753082469136$; $1975308 + 2469136 = 4444444$
$4927941^2= 24284602499481$; $2428460 + 2499481 = 4927941$
$5072059^2=25725782499481$; $2572578 + 2499481 = 5072059$
$5555556^2=30864202469136$; $3086420 + 2469136 = 5555556$
$9372385^2=87841600588225$; $8784160 + 0588225 = 9372385$
$9999999^2=99999980000001$; $9999998 + 0000001 = 9999999$
$11111112^2=123456809876544$; $1234568 + 9876544 = 11111112$

$13641364^2 = 186086811780496; 01860868 + 11780496 = 13641364$
$16590564^2 = 275246813838096; 02752468 + 13838096 = 16590564$
$19273023^2 = 371449415558529; 03714494 + 15558529 = 19273023$
$19773073^2 = 390974415863329; 03909744 + 15863329 = 19773073$
$24752475^2 = 612685018625625; 06126850 + 18625625 = 24752475$
$25252525^2 = 637690018875625; 06376900 + 18875625 = 25252525$
$30884184^2 = 953832821345856; 09538328+21345856=30884184$
$36363636^2 = 1322314023140496; 13223140 + 23140496 = 36363636$
$38883889^2 = 1511956823764321; 15119568 + 23764321 = 38883889$
$44363341^2 = 1968106024682281; 19681060 + 24682281 = 44363341$
$44525548^2 = 1982524424700304; 19825244 + 24700304 = 44525548$
$49995000^2 = 2499500025000000; 24995000 + 25000000 = 49995000$
$50005000^2 = 2500500025000000; 25005000 + 25000000 = 50005000$
$55474452^2 = 3077414824700304; 30774148 + 24700304 = 55474452$
$55636659^2 = 3095437824682281; 30954378 + 24682281 = 55636659$
$61116111^2 = 3735179023764321; 37351790 + 23764321 = 61116111$
$63636364^2 = 4049586823140496; 40495868 + 23140496 = 63636364$
$69115816^2 = 4776996021345856; 47769960 + 21345856 = 69115816$
$74747475^2 = 5587185018875625; 55871850 + 18875625 = 74747475$
$75247525^2 = 5662190018625625; 56621900 + 18625625 = 75247525$
$80226927^2 = 6436359815863329; 64363598 + 15863329 = 80226927$
$80726977^2 = 6516844815558529; 65168448 + 15558529 = 80726977$
$83409436^2 = 6957134013838096; 69571340 + 13838096 = 83409436$
$86358636^2 = 7457814011780496; 74578140 + 11780496 = 86358636$
$88888888^2 = 7901234409876544; 79012344 + 09876544 = 88888888$
$91838088^2 = 8434234407495744; 84342344 + 07495744 = 91838088$
$94520547^2 = 8934133805179209; 89341338 + 05179209 = 94520547$
$99999999^2 = 9999999800000001; 99999998 + 00000001 = 99999999$
$234567901^2 = 55022100179545801; 055022100 + 179545801 = 234567901$
$332999667^2 = 110888778222110889; 110888778 + 222110889 = 332999667$
$432432432^2 = 186997808245434624; 186997808 + 245434624 = 432432432$
$567567568^2 = 322132944245434624; 322132944 + 245434624 = 567567568$
$667000333^2 = 444889444222110889; 444889444 + 222110889 = 667000333$
$765432099^2 = 585886298179545801; 585886298 + 179545801 = 765432099$
$999999999^2 = 999999998000000001; 999999998 + 000000001 = 999999999$
$1111111111^2 = 1234567900987654321; 123456790 + 0987654321 = 1111111111$
$1776299581^2 = 3155240201460775561; 0315524020 + 1460775561 = 1776299581$
$2020202020^2 = 4081216201612080400; 0408121620 + 1612080400 = 2020202020$

$3846956652^2 = 14799075482367049104; 1479907548 + 2367049104 = 3846956652$

$3888938889^2 = 15123845682376554321; 1512384568 + 2376554321 = 3888938889$

$4090859091^2 = 16735128102417346281; 1673512810 + 2417346281 = 4090859091$

$4132841328^2 = 17080377442424803584; 1708037744 + 2424803584 = 4132841328$

....

3. *Generation of Kaprekar Numbers*

For each integer, $N > 1$, let $x(N)$ denote a set of positive integers X for which there exists integers A and B such that

$$X^2 = AN + B, \text{ where } 0 < B < N \tag{3}$$

$$X = A + B \tag{4}$$

Using (4) in (3), we get

$$X^2 = AN + X - A$$

or $$X(X-1) = A(N-1) \tag{5}$$

From above equation, we have $1 \leq X \leq N - 1$.

For some positive integers d and d', suppose

$$dd' = N - 1 \text{ and } (d,d') = 1 \tag{6}$$

The above equation shows that d is a unitary divisor of $(N - 1)$ since $(d,d') = 1$.

Take $$X' = N - X \tag{7}$$

Let $$X = dm \text{ and } X' = d'm' \tag{8}$$

Where m and m' are positive integers.

Using (6), (7) & (8), we obtain

$$dm + d'm' = N = dd' + 1 \tag{9}$$

Solving the above equation, we obtain

$$(dm)^2 = N^2 + (d'm')^2 - 2Nd'm'$$
$$= N^2 + (d'm')^2 - Nd'm' - Nd'm'$$
$$= N^2 + (d'm')^2 - Nd'm' - (dm + d'm')d'm'$$
$$= N^2 - Nd'm' - mm'dd'$$
$$= N^2 - Nd'm' - mm'(N-1)$$

Since $X = dm$, we can write the above expression as

$$X^2 = (N - d'm' - mm')N + mm' \tag{10}$$

From (8) and (9), we get

$$X = dm = N - d'm'$$

or $$X = (N - d'm' - mm') + mm' \tag{11}$$

Comparing (10) with (3) and (11) with (4), we have

$$A = N - d'm' - mm' \tag{12}$$

$$B = mm' \tag{13}$$

Using both (12) and (13) in (10) and (11), we obtain

$$X^2 = AN + B \tag{14}$$

$$X = A + B \tag{15}$$

The Equation (14) is same as (1) for $N = b^n$ and Equation (15) is same as (2). Hence, Kaprekar numbers X can be generated using equations (6), (8) and (9) taking $b = 10$. The following Table-1 shows the generation of n-Kaprekar numbers X for $1 \leq n \leq 5$ for the unitary divisors d of $(N - 1) = 10^n - 1$ as per equation (6). Using (12) and (13), this table also gives the values of A and B, whose sum is equal to the Kaprekar number X.

Table -1: Generation of all n - Kaprekar numbers, $1 \leq n \leq 5$.

n	d'	$d = \frac{10^n - 1}{d'}$ Eqn. (6)	m Eqn.(9)	m' Eqn.(9)	$X = dm$ Eqn.(8)	A $=(10^n - d'm' - mm')$ Eqn.(12)	$B = mm'$ Eqn.(13)
1	9	1	1	1	1	0	1
	1	9	1	1	9	8	1
2	99	1	1	1	1	0	1
	11	9	5	5	45	20	25
	9	11	5	5	55	30	25
	1	99	1	1	99	98	1
3	999	1	1	1	1	0	1
	37	27	11	19	297	88	209
	27	37	19	11	703	494	209
	1	999	1	1	999	998	1
4	9999	1	1	1	1	0	1
	1111	9	247	7	2223	494	1729
	909	11	248	8	2728	744	1984
	101	99	50	50	4950	2450	2500
	99	101	50	50	5050	2550	2500
	11	909	8	248	7272	5288	1984
	9	1111	7	247	7777	6048	1729
	1	9999	1	1	9999	9998	1
5	99999	1	1	1	1	0	1
	11111	9	8642	2	77778	60494	17284
	2439	41	119	39	4879	238	4641
	271	369	224	64	82656	68320	14336
	369	271	64	224	17344	3008	14336
	41	2439	39	119	95121	90480	4641
	9	11111	2	8642	22222	4938	17284
	1	99999	1	1	99999	99998	1

The values of X,A and B obtained in Table-1 are same as the values given in Section-2.

4. Some Properties of Kaprekar Numbers :

The following are some properties of Kaprekar numbers.

(a) If X is n - Kaprekar number, then $X^2 \equiv X \,(mod\; 10^n - 1)$.

Proof: According to equation (1) for = 10, we have

$$X^2 = A \times 10^n + B$$

$$= A \times (10^n - 1 + 1) + B$$
$$= A \times (10^n - 1) + A + B$$

Using (2), we get

$$X^2 = A\ (10^n - 1) + B$$
$$\Rightarrow X^2 \equiv X\ (mod\ 10^n - 1) \quad (16)$$

For example, if $X = 55$, then $X^2 = 3025$. When 3025 is divided by $10^n - 1 = 10^2 - 1 = 99$, the remainder is $55 = X$.

(b) If $X \equiv t\ (mod\ 9)$, then $X^2 \equiv t^2\ (mod\ 9)$ and $t = 0\ or\ 1$,

Proof: For integers, X, s and t, we can write

$$X = 9 \times s + t \quad (17)$$
$$\text{or} \quad X \equiv t\ (mod\ 9) \quad (18)$$

When $X = 45$ is divided by 9, the remainder $t = 0$. If $X = 55$ is divided by 9, the remainder $t = 1$.
Squaring (17), we get

$$X^2 = 9^2 \times s^2 + 2 \times 9 \times s + t^2$$
$$\Rightarrow X^2 \equiv t^2\ (mod\ 9) \quad (19)$$

For example, if $X = 45$, then $X^2 = 2025$. When 2025 is divided 9, the remainder is $0 = t^2$.
Similarly, if $X = 55$, then $X^2 = 3025$. When 3025 is divided 9, the remainder is $1 = t^2$.
Hence, $t = 0\ or\ 1$.

(c) *Case 1*: For odd n - Kaprekar numbers X, A is even and B is odd.
Proof: Let X is odd and greater than 1. From (2), we have

$$X = A + B$$

Since X is odd, either A could be odd and B even, or A could be even and B odd. From (1), we have

$$X^2 = A \times 10^n + B$$

If X is odd, it follows that X^2 is odd. As $n \geq 1$, $A \times 10^n$ is even. So B must be odd from the above equation. Hence, from (2), A is even if X is odd. For example, if $X = 703$, we have $A = 494$ and $B = 209$.

Case 2: For even n - Kaprekar numbers X, A is even and B is even.

Proof: Let X is even and from (2), we have

$$X = A + B$$

Since X is even, either both A and B could be even, or both A and B could be odd. From (1), we have

$$X^2 = A \times 10^n + B$$

If X is even, it follows that X^2 is even. As $n \geq 1$, $A \times 10^n$ is even. So B must be even from the above equation. Hence, from (2), A is even if X is even. For example, if $X = 2728$, we have $A = 744$ and $B = 1984$.

(d) The maximum value n_{max} of n in (1) is $2 \log_{10} X$.
Proof: From (1), we obtain

$$10^n = \frac{x^2 - B}{A}$$
$$\Rightarrow \quad n = \log_{10} \left\lfloor \frac{x^2 - B}{A} \right\rfloor$$

For $B > 0$ and $A > 0$, we get $n < \log_{10} X^2$ or $n < 2 \log_{10} X$.
Hence, $n_{max} = 2 \log_{10} X$

(e) The upper bound of B is $\dfrac{10^{2n}}{4(10^{n-1})}$.

Proof: Substituting (2) in (1), we get

$$(A + B)^2 = A \times 10^n + B$$

$$\Rightarrow A^2 + A(2B - 10^n) + (B^2 - B) = 0$$

This is a quadratic equation in A and the roots of A are

$$A = \frac{-(2B - 10^n) \pm \sqrt{(2B - 10^n)^2 - 4(B^2 - B)}}{2}$$

$$\Rightarrow A = \frac{10^n - 2B \pm \sqrt{10^{2n} - 4B(10^n - 1)}}{2}$$

Since, A is an integer and real, we have $[10^{2n} - 4B(10^n - 1)]$ is a perfect square and

$$10^{2n} \geq 4B(10^n - 1)$$

$$\Rightarrow B \leq \frac{10^{2n}}{4(10^n - 1)}$$

Hence, the upper bound of B is $\dfrac{10^{2n}}{4(10^n - 1)}$.

(f) 0, 10, 100, 1000, 10000, ..., etc. are not Kaprekar numbers, because B is zero for these numbers and it should not be zero as per the definition for Kaprekar numbers.

$$0^2 = 0; 0 + 0 = 0$$

$$10^2 = 100; 10 + 0 = 10$$

$$100^2 = 10000; 100 + 00 = 100 \text{, etc.}$$

(g) For any base b, there exist infinite Kaprekar numbers of the form $(b^n - 1)$, where $n = 1,2,3, \ldots$, etc. For $b = 10$, we have the Kaprekar numbers 9,99,999, 9999,99999, ..., etc.

(h) If $(b^x - 1)$ is a Kaprekar number in base b, then $(b^x - 1)^2 = b^x [(b^x - 1) - 1] + 1$. This identity can be verified by taking $b = 10$ and $x = 3$. For these values, $(b^x - 1)^2 = b^x [(b^x - 1) - 1] + 1$ 998001.

(i) If we take any Kaprekar number X with p digits, then $(X^2 - X)$ will be divisible by $(10^p - 1)$, i.e., by numbers like 9,99, 999, ... etc. For example, if $X = 45$, then $p = 2$ and $X^2 - X = 1980$, which is divisible by $(10^2 - 1) = 99$.

(j) A Kaprekar number X in a base b can be expressed as

$$X^2 = \sum_{i=0}^{k-1} d_i b^i \tag{20}$$

Where, k represents the number of digits in X^2 and d_i is the ith digit in X^2 with d_0 the least significant digit and d_{k-1} the most significant digit.

Example: Let $X = 297$, then $X^2 = 88209 = 88 \times 10^3 + 209$

Therefore, $k = 5, b = 10, d_0 = 9, d_1 = 0, d_2 = 2, d_3 = 8$ & $d_4 = 8$.

$$\sum_{i=0}^{4} d_i b^i = d_0 b^0 + d_1 b^1 + d_2 b^2 + d_3 b^3 + d_4 b^4$$

$$= 9 \times 10^0 + 0 \times 10^1 + 2 \times 10^2 + 8 \times 10^3 + 8 \times 10^4$$

$$= 9 + 0 + 200 + 8000 + 80000 = 88209 = X^2$$

Hence, (20) is proved.

(k) The Kaprekar numbers X occur in complementary pairs whose sum is 10^n. For example, there are two pairs (1,999) and (297, 703) of Kaprekar numbers for $n = 3$ (Table- 1). The sum of each pair is 10^3.

(l) If is n - Kaprekar number, then

$$X^3 = P \times 10^n - Q \tag{21}$$

$$X = P - Q \tag{22}$$

Subtracting (22) from (21), we get

$$X^3 - X = P(10^n - 1)$$

$$\Rightarrow P = \frac{X^3 - X}{10^n - 1} \tag{23}$$

P can be calculated from (23).

Examples:

- If $X = 55$ for $n = 2$ (Table-1), we have from (23)

  $$P = \frac{55^3 - 55}{99} = 1680$$

  $55^3 = 166375 = 1680 \times 10^2 - 1625$ and $55 = 1680 - 1625$

  Hence, $Q = 1625$

- If $X = 703$ for $n = 3$ (Table-1), we get from (23)

  $$P = \frac{703^3 - 703}{999} = 347776$$

  $703^3 = 347428927 = 347776 \times 10^3 - 347073$ and

  $703 = 347776 - 347073$. That is, $Q = 347073$

5. Kaprekr Routine and Kaprekar Constants :

The Kaprekar routine is an algorithm discovered in 1949 by D.R. Kaprekar for 4-digit numbers, but it can be generalized to n - digit numbers. It is not applicable for single digit number. To apply the Kaprekar routine, choose a four digit number, where all the digits are not same. Arrange the digits to get the largest and smallest numbers. Subtract the smallest number from the largest number to get a new number. Repeat the same operation for each new number. After few steps, we reach the number 6174. When we reach 6174, the operation repeats itself, returning 6174 every time. The number 6174 is known as *kernel* of this operation. Also this number is called the *Kaprekar constant* for four digit numbers. A number that remains unchanged on applying Kaprekar routine on it is known as Kaprekar constant.The number of steps or iterations required to reach the Kaprekar constant 6174 in case of 4 digit numbers cannot be greater than 7.

Examples:

(a) Consider the number 2005. The maximum number we can make with these digits is 5200 and the minimum is 0025 or 25. The subtractions are

5200 – 0025 = 5175
7551 – 1557 = 5994
9954 – 4599 = 5355
5553 – 3555 = 1998
9981 – 1899 = 8082

8820 – 0288 = 8532
8532 – 2358 = 6174
7641 – 1467 = 6174

In the above operation, the number 6174 is reached in 7 steps or iterations.

(b) In case of the number 1000, the number of iterations is five to reach the number 6174.
1000 – 0001 = 0999
9990 – 0999 = 8991
9981 – 1899 = 8082
8820 – 0288 = 8532
8532 – 2358 = 6174

(c) Take the number 5644. The following three steps or iterations are taken for getting Kaprekar constant 6174.
6544 – 4456 = 2088
8820 – 0288 = 8532
8532 – 2358 = 6174

(d) Take the number 1746. In this case, Kaprekar constant is arrived in one step or iteration.
7641 – 1467 = 6174

Let us now apply Kaprekar operation to three digit numbers. The Kaprekar constant for three digit numbers is 495.

Examples:

(e) Consider the number 753. The Kaprekar constant 495 is reached in 3 steps.
753–357 = 396
963 – 369 = 594
954 – 459 = 495
954 – 459 = 495

(f) Similarly applying the Kaprekar operation to the number 297, we have
972 – 279 = 693
963 – 369 = 594
954 – 459 = 495

We have seen that four and three digit numbers reach a unique kernel. But there is no unique kernel for two digit numbers.

Examples:

(g) Applying Kaprekar operation to the number 28, we get
82 – 28 = 54
54 – 45 = 09
90 – 09 = 81
81 – 18 = 63
63 – 36 = 27
72 – 27 = 45
54 – 45 = 09

(h) Take the number 62.

62 – 26 = 36
63 – 36 = 27
72 – 27 = 45
54 – 45 = 09
90 – 09 = 81
81 – 18 = 63
63 – 36 = 27
72 – 27 = 45
54 – 45 = 09

The two digit numbers are giving the loop 9 → 81→63→27→45→9, but the kernel is not reached. Similarly, there is no kernel for five digit numbers. But all five digit numbers give one of the following loops by Kaprekar operation.

- 71973→83952→74943→62964→71973
- 75933→63954→61974→82962→75933
- 59994→53955→59994

Examples:

(i) Consider the five digit number 35768.

87653 – 35678 = 51975
97551 – 15579 = 81972
98721 – 12789 = 85932
98532 – 23589 = 74943
97443 – 34479 = 62964
96642 – 24669 = 71973
97731 – 13779 = 83952
98532 – 23589 = 74943
97443 – 34479 = 62964
96642 – 24669 = 71973

(j) Take the number 21549.

95421 – 12459 = 82962
98622 – 22689 = 75933
97533 – 33579 = 63954
96543 – 34569 = 61974
97641 – 14679 = 82962
98622 – 22689 = 75933

(k) Considering the number 17344, we get the following steps.

74431 – 14437 = 59994
99954 – 45999 = 53955
95553 – 35559 = 59994

The kernel values are shown in the Table-2. For three and four digit numbers, we get a unique value for the kernel. In case of numbers having 6, 8 and 9 digits, there are two kernel values. For the number with 10 digits, there will be 3 kernel values. For 2 digit, 5 digit and 7 digit numbers, the sequence obtained by Kaprekar operation enters a loop.

Table-2: Kernel values

No of digits in the number	Kernel or Kaprekar constant values
2	None
3	495 (Unique value)
4	6174 (Unique value)
5	None
6	549945, 631764
7	None
8	63317664, 97508421
9	554999445, 864197532
10	6333176664, 9753086421, 9975084201

Let us now discuss about the frequency of iterations to reach the Kaprekar constant 6174 in case of four digit numbers. Let us consider the 8991 four digit numbers from 1000 to 9999, excluding those nine numbers with four equal digits (1111, 2222,3333,4444,5555,6666,7777,8888 & 9999). In case of the four digit number 6174, without performing any Kaprekar operation, 6174 has already been reached. So, in this case the number of iterations is zero and the frequency of iteration is one. There are 356 four digit numbers among 8991 four digit numbers which reach the number 6174 after1 iteration. Hence, the frequency of iterations is 356. Similarly, there are 519 four digit numbers which reach 6174 after 2 iterations. So, the frequency of iterations is 519. Table-3 shows the number of iterations and frequency of iterations needed to reach 6174 for all the 8991 four digit numbers.

Table -3: No. of iterations and frequency needed to reach 6174

No. of Iterations	Frequency
0	1
1	356
2	519
3	2124
4	1124
5	1379
6	1508
7	1980
	Total = 8991

6. Kaprekar Numbers with Binary Base 2 :

0 and 1 are called binary numbers. The basic rules of binary addition are 0 + 0 = 0; 0 + 1 = 1; 1 + 0 = 1; 1 + 1 = 10; 10 + 1 = 11; 11 + 1 = 100. The rules for binary multiplication are 1× 0 = 0, 0×1 = 0 and 1× 1 = 1. The following calculations using the rules of binary multiplication and addition for the 7 binary numbers show that the square of a binary number can be split up into two parts, which add up to give the original binary number. Hence these binary numbers represent Kaprekar numbers with binary base 2.

(a) $1^2 = 1 \times 1 = 1 = 01; 0 + 1$

(b) $11^2 = 11 \times 11 = 1001; 10 + 01 = 11$

(c) $110^2 = 110 \times 110 = 100100; 10 + 0100 = 0110 = 110$

(d) $111^2 = 111 \times 111 = 110001; 110 + 001 = 111$

(e) $1010^2 = 1010 \times 1010 = 1100100; 110 + 0100 = 1010$

(f) $1111^2 = 1111 \times 1111 = 11100001; 1110 + 0001 = 1111$

(g) $11100^2 = 11100 \times 11100 = 1100010000; 1100 + 010000 = 011100 = 11100$

The first 26 Kaprekar numbers in binary form are
1, 11, 110, 111, 1010, 1111, 11100, 11111, 100100, 110011, 111111, 1010101, 1011011, 1111000, 1111111, 10001000, 10010011, 10101011, 10111011, 11001101, 11111111, 101010110, 101011111, 101101101, 111110000, 111111111, ...

7. Conclusion

In this article, Kaprekar numbers and their properties are given. The Kaprekar constants are calculated by applying kaprekar routine. Kaprekar numbers are also expressed in binary form.

8. References

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