Ultra Scientist Vol. 28(4)A, 213-222 (2016). ISSN 2231-346X (Print)
www.ultrascientist.org 2319-8044 (Online)

Smarandache-Alpha Level Subgroups

R. GOWRI [1] and T. RAJESWARI [2]

[1] Department of Mathematics,
Government College for Women (Autonomous), Kumbakonam (India)
[2] Research Scholar, Department of Mathematics,
Government College for Women(Autonomous), Kumbakonam (India)
E-mail: rajeswari.mrt24@gmail.com

(Acceptance Date 30th July, 2016, Online Publication Date 2nd Sep. 2016)

Abstract

The concept of S-α level subgroup of an S-α fuzzy semigroup is defined and its characterizations are obtained. It is also discussed about S-α fuzzy semigroups relative to a finite cyclic group.

Key words: S-Semigroup,fuzzy group, α-fuzzy set, S-α fuzzy semigroup, level subgroups.

Mathematical Subject Classification: 03E72, 08A72, 20N25.

1 Introduction and Preliminaries

In 1965, Zadeh introduced the notion of fuzzy set[1]. In 1971, A. Rosenfeld defined fuzzy groups[2] and many group theory results have been extended to fuzzy groups. P.Sivaramakrishna Das introduced the concept of level subgroups of a fuzzy group[3]. W.B.Vasantha Kandasamy studied about Smarandache fuzzy semigroups in[5]. P.K.Sharma introduced the notion of α-fuzzy set, α-fuzzy group, α-fuzzy coset and analysed their characterizations in[6]. R. Gowri and T. Rajeswari introduced the concept of S-α fuzzy semigroup, S-α fuzzy left, right cosets and S-α fuzzy normal subsemigroup and obtained their properties[7]. In this paper, S-α level subgroups of an S-α fuzzy semigroup is defined and its characterizations are obtained. It is also discussed about S-α fuzzy semigroups relative to a finite cyclic group.

Throughout this paper, α denotes a member of $[0,1]$.

Definition 1.1 Let X be a non empty set. A **fuzzy subset** A of X is a function $A : X \to [0,1]$.

Definition 1.2 A fuzzy subset A of a group G is called a **fuzzy subgroup** of G if

(i) $A(xy) \geq min\{A(x), A(y)\}$

(ii) $A(x^{-1}) = A(x)$, for all $x, y \in G$.

Definition 1.3 Let A be a fuzzy subset of a set S. For $t \in [0,1]$, the set $A_t = \{x \in S \mid A(x) \geq t\}$ is called a **level subset** of A.

Definition 1.4 Let G be a group and A be a fuzzy subgroup of G. The subgroups $A_t, t \in [0,1]$ and $t \leq A(e)$, are called **level subgroups** of A.

Definition 1.5 A semigroup S is said to be a **Smarandache semigroup** (S-semigroup) if there exists a proper subset P of S which is a group under the same binary operation in S.

Definition 1.6 Let A be a fuzzy subset of a group G. Let $\alpha \in [0,1]$. Then an **α-fuzzy subset of G (with respect to a fuzzy set A)**, denoted by A^{α}, is defined as $A^{\alpha}(x) = min\{A(x), \alpha\}$, for all $x \in G$.

Definition 1.7 Let G be an S-semigroup. Let A be a fuzzy subset of G and let $\alpha \in [0,1]$. A is called a **Smarandache-α fuzzy semigroup (S-α fuzzy semigroup)** if there exists a proper subset P of G which is a group and the restriction of A to $P(A_P : P \to [0,1])$ is such that A_P^{α} is a fuzzy group.
That is,

(i) $A_P^{\alpha}(xy) \geq min\{A_P^{\alpha}(x), A_P^{\alpha}(y)\}$

(ii) $A_P^{\alpha}(x^{-1}) = A_P^{\alpha}(x)$, for all $x, y \in P$.

Result[7] *1.8* If $A : G \to [0,1]$ is an S-α fuzzy semigroup of an S-semigroup G relative to a group P which is a proper subset of G, then

(i) $A_P^{\alpha}(x) \leq A_P^{\alpha}(e)$, where e is the identity element of P.

(ii) $A_P^{\alpha}(xy^{-1}) = A_P^{\alpha}(e) \Rightarrow A_P^{\alpha}(x) = A_P^{\alpha}(y)$, for all $x, y \in P$.

2 Smarandache-Alpha Level Subgroups:

In this section the concept of S-α level subgroup of an S-α fuzzy semigroup is defined and its characterizations are obtained. We also discuss about S-α fuzzy semigroups relative to a finite cyclic group.

Definition 2.1 Let G be an S-semigroup. Let $A : G \to [0,1]$ be a fuzzy subset of G. For $t \in [0,1]$, a **Smarandache-alpha level subset (S-α level subset)** of the fuzzy subset A, denoted by $A^{\alpha}_{P_t}$, is defined as $A^{\alpha}_{P_t} = \{x \in P / A^{\alpha}_P(x) \geq t\}$, where P is a proper subset of G which is a group.

Example 2.2 Consider $S(3)$ which is an S-semigroup. Let $A: S(3) \to [0,1]$ be defined as

$$A(x) = \begin{cases} 0.8, & \text{if } x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\ 0.6, & \text{if } x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\ 0.7, & \text{otherwise} \end{cases}$$

Let $P = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$.

Clearly P is a Proper subset of $S(3)$ and is also a group. Let $\alpha = 0.75$ and $t = 0.65$.

Then $A^{\alpha}_{P_t} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\}$ is an S-α level subset of A.

Theorem 2.3 Let A be an S-α fuzzy semigroup of an S-semigroup G relative to a group P in G. If $t \in [0,1]$ and $t \leq A^{\alpha}_P(e)$, then the S-α level subset $A^{\alpha}_{P_t}$ is a subgroup of P, where e is the identity of P.

Proof : By assumption, $A^{\alpha}_{P_t} = \{x \in P / A^{\alpha}_P(x) \geq t\}$ is a non empty subset of P. If $x, y \in A^{\alpha}_{P_t}$, then $min\{A^{\alpha}_P(x), A^{\alpha}_P(y)\} \geq t$. Since A^{α}_P is a fuzzy group, $A^{\alpha}_P(xy) \geq t$ which implies that $xy \in A^{\alpha}_{P_t}$. Also $A^{\alpha}_P(x^{-1}) = A^{\alpha}_P(x)$. Since $x \in A^{\alpha}_{P_t}$, $A^{\alpha}_P(x) \geq t$ which implies that $A^{\alpha}_P(x^{-1}) \geq t$. Therefore $x^{-1} \in A^{\alpha}_{P_t}$. Thus $A^{\alpha}_{P_t}$ is a subgroup of P. □

Theorem 2.4 Let G be an S-semigroup and $P \subset G$ be a group. Let A be a fuzzy subset of G such that $A^{\alpha}_{P_t}$ is a subgroup of P, for all $t \in [0,1]$ and $t \leq A^{\alpha}_P(e)$. Then A is an S-α fuzzy semigroup of G.

Proof: Let $a, b \in P$ and let $A^{\alpha}_P(a) = t_1$ and $A^{\alpha}_P(b) = t_2$. Therefore $a \in A^{\alpha}_{P_{t_1}}$ and $b \in A^{\alpha}_{P_{t_2}}$. Suppose that $t_1 < t_2$. Then $A^{\alpha}_{P_{t_2}} \subset A^{\alpha}_{P_{t_1}}$ which implies that $b \in A^{\alpha}_{P_{t_1}}$. Since $a, b \in A^{\alpha}_{P_{t_1}}$ and $A^{\alpha}_{P_{t_1}}$ is a subgroup of P, $ab \in A^{\alpha}_{P_{t_1}}$. Therefore $A^{\alpha}_P(ab) \geq t_1 = min\{A^{\alpha}_P(a), A^{\alpha}_P(b)\}$. Now let $a \in P$ and $A^{\alpha}_P(a) = t$. Therefore $a \in A^{\alpha}_{P_t}$ which implies that $a^{-1} \in A^{\alpha}_{P_t}$, since $A^{\alpha}_{P_t}$ is a subgroup of P. Then $A^{\alpha}_P(a^{-1}) \geq t = A^{\alpha}_P(a)$. Thus A^{α}_P is a fuzzy group and hence A is an S-α fuzzy semigroup of G. □

Definition 2.5 Let A be an S-α fuzzy semigroup of an S-semigroup G relative to a group P in G. If $t \in [0,1]$ and $t \leq A_P^\alpha(e)$, then the subgroups $A_{P_t}^\alpha$ are said to be **Smarandache-alpha level subgroups (S-α level subgroups)** of A with respect to P.

Example2.6 Let $G = \{e, a, b, c, d, e, f, g\}$ which is a semigroup by the following table.

0	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	e	c	b	a	a	a
b	b	c	e	a	b	b	b
c	c	b	a	e	c	c	c
d	d	a	a	a	d	f	g
f	f	b	b	b	d	f	g
g	g	c	c	c	d	f	g

Let $P = \{e, a, b, c\}$ be the klein four group. Let a fuzzy subset A of G be defined as

$$A(x) = \begin{cases} 1, & if \ \ x = e, a \\ 3/4, & if \ \ x = b, c \\ 0, & otherwise \end{cases}$$

If $\alpha = 0.85$, then A_P^α is an S-α fuzzy semigroup of G. If $t = 0.8$, then $A_{P_t}^\alpha = \{e, a\}$ which is a subgroup of P and is an S-α level subgroup of A with respect to P

Remark 2.7 If G is an S-semigroup and $P \subset G$ is a group which is also finite, then the number of subgroups of P is finite whereas the number of S-α level subgroups of an S-α fuzzy semigroup A of G with respect to P seems to be infinite. But since every S-α level subgroup relative to P is a subgroup of P, not all these S-α level subgroups will be distinct. This is characterised in the following theorem.

Theorem 2.8 Let A be an S-α fuzzy semigroup of an S-semigroup G relative to a group P in G. Let $t_1, t_2 \in [0,1]$ and $t_1 < t_2$. Then two S-α level subgroups $A_{P_{t_1}}^\alpha, A_{P_{t_2}}^\alpha$ of A with respect to P are equal if and only if there is no $x \in P$ such that $t_1 < A_P^\alpha(x) < t_2$.

Proof : Assume that $A_{P_{t_1}}^\alpha = A_{P_{t_2}}^\alpha$. Suppose that there exists $x \in P$ such that $t_1 < A_P^\alpha(x) < t_2$. Then $x \in A_{P_{t_1}}^\alpha$ and $x \notin A_{P_{t_2}}^\alpha$. This implies that $A_{P_{t_1}}^\alpha \neq A_{P_{t_2}}^\alpha$ which is a contradiction. Conversely, assume that there is no $x \in P$ such that $t_1 < A_P^\alpha(x) < t_2$. Since $t_1 < t_2$, $A_{P_{t_2}}^\alpha \subset A_{P_{t_1}}^\alpha$. If $x \in A_{P_{t_1}}^\alpha$, then $t_1 \leq A_P^\alpha(x)$. Also $A_P^\alpha(x)$ does not lie between t_1 and t_2. Therefore $t_2 \leq A_P^\alpha(x)$ which implies $x \in A_{P_{t_2}}^\alpha$. Thus $A_{P_{t_1}}^\alpha = A_{P_{t_2}}^\alpha$. □

Corollary 2.9 Let A be an S-α fuzzy semigroup of an S-semigroup G relative to a finite group P. Let $Im(A_P^\alpha) = \{t_i / A_P^\alpha(x) = t_i$, for some $x \in P\}$. Then $A_{P_{t_i}}^\alpha$'s are the only S-α level subgroups of A with respect to P.

Proof : Since $t_i \leq A_P^\alpha(e)$, by theorem 2.3, $A_{P_{t_i}}^\alpha$ is a subgroup of P for each i. Let $t \in [0,1]$. If $t = A_P^\alpha(e)$, then $A_{P_t}^\alpha = \{e\}$ which is an S-α level subgroup of A. Let $t \neq Im(A_P^\alpha)$. Suppose that $t_i < t < t_j$, where $t_i, t_j \in Im(A_P^\alpha)$. Since $A_P^\alpha(x) \neq t$, $x \in P$, by theorem 2.8, $A_{P_{t_i}}^\alpha = A_{P_{t_j}}^\alpha = A_{P_t}^\alpha$ which is an S-α level subgroup of A with respect to P. Suppose that $t < t_r$, where t_r is the least element in $Im(A_P^\alpha)$. Then $A_P^\alpha(x) \geq t_r$, for all $x \in P$. Therefore $A_{P_{t_r}}^\alpha = P = A_{P_t}^\alpha$. Thus for any $t \in [0,1]$, $A_{P_{t_i}}^\alpha$'s are the only S-α level subgroups of A with respect to P. □

Theorem 2.10 Let G be an S-semigroup. Then any subgroup H of a group P in G can be realised as an S-α level subgroup of some S-α fuzzy semigroup of G relative to P.

Proof : $P \subset G$ is a group and H is a subgroup of P. Let $A : G \to [0,1]$ be defined as

$$A(x) = \begin{cases} t, & if\ x \in H \\ 0, & if\ x \notin H, 0 < t < 1 \end{cases}$$

Then $A_P : P \to [0,1]$ is such that

$$A_P(x) = \begin{cases} t, & if\ x \in H \\ 0, & if\ x \notin H, 0 < t < 1 \end{cases}$$

Let $\alpha \geq t$. Let $x, y \in P$. Case(i): Suppose $x, y \in H$. Then $xy \in H$ and hence $A_P(xy) = t$ and $A_P(x) = A_P(y) = t$. Since $\alpha \geq t$, $A_P^\alpha(xy)$ $= min\{A_P(xy), \alpha\} = t = min\{A_P^\alpha(x), A_P^\alpha(y)\}$. Case (ii): Suppose $x \in H$ and $y \notin H$. Then $xy \notin H$. Otherwise, $y \in H$ which is a contradiction. Therefore $A_P(x) = t, A_P(y) = 0$ and $A_P(xy) = 0$. Then $A_P^\alpha(xy) = min\{0, \alpha\}$ $= min\{A_P^\alpha(x), A_P^\alpha(y)\}$. Case (iii): Suppose both $x, y \notin H$. Then $xy \in H$ or $xy \notin H$. If $xy \in H$, then $A_P^\alpha(xy) = min\{t, \alpha\} = t \geq$ $min\{A_P^\alpha(x), A_P^\alpha(y)\}$. If $xy \notin H$, then $A_P^\alpha(xy) = min\{0, \alpha\} = min\{A_P^\alpha(x), A_P^\alpha(y)\}$ In all the three cases, we have $A_P^\alpha(xy) \geq$ $min\{A_P^\alpha(x), A_P^\alpha(y)\}$. If $x \in H$, then $x^{-1} \in H$. Therefore $A_P(x) = t = A_P(x^{-1})$ which implies $A_P^\alpha(x) = A_P^\alpha(x^{-1})$. If $x \in H, x^{-1} \in H$ or $x^{-1} \notin H$. If $x^{-1} \in H$, then $A_P^\alpha(x^{-1}) = min\{t, \alpha\} = t \geq 0$ $= A_P^\alpha(x)$. If $x^{-1} \notin H$, then $A_P^\alpha(x^{-1}) = 0 = A_P^\alpha(x)$.

Thus A_P^α is a fuzzy group and hence A is an S-α fuzzy semigroup of G. Moreover , since $\alpha \geq t$, $A_{P_t}^\alpha = \{x \in P / A_P^\alpha(x) \geq t\} = H$. □

Theorem 2.11 Let G be an S-semigroup. Let $\overline{A}$ be the collection of all S-α fuzzy semigroups of G relative to a group P in G and let $\overline{B}$ be the collection of all S-α level subgroups of members of $\overline{A}$ with respect to P. Then there is a one-to-one correspondence between the subgroups of P and the equivalence class of S-α level subgroups with respect to P under a suitable equivalence relation defined on $\overline{B}$.

Proof : If we take $I = [0,1]$, then an element of $\overline{A}$ together with an element of I will produce an element in $\overline{B}$. Thus $\overline{B} = \overline{A} \times I$. Now let us define a relation ~ on $\overline{B}$ by (f_P^α, t) ~ (g_t^α, t') if and only if $f_{P_t}^\alpha = g_{P_{t'}}^\alpha$. It is easy to see that ~ is an equivalence relation on $\overline{B}$ and hence $\overline{B}$ can be expressed as a union of disjoint equivalence classes of S-α level subgroups with respect to P. Let us define a function Ψ from the collection of all the equivalence classes of S-α level subgroups of members of $\overline{A}$ with respect to P and the collection of all the subgroups of P as $\Psi([f_{P_t}^\alpha]) = f_{P_t}^\alpha$, where $f_{P_t}^\alpha$ is a subgroup of P. Since (by theorem 2.10) every subgroup H of P will be equal to $f_{P_t}^\alpha$ for some $f \in \overline{A}$ and $t \in [0,1]$, it can be shown that Ψ will be both 1-1 and onto. □

Theorem 2.12 Let G be an S-semigroup relative to a cyclic group P of prime order p. Let μ be a fuzzy subset of G. Then μ is an S-α fuzzy semigroup of G relative to $P \Leftrightarrow \mu_P^\alpha(x) = \mu_P^\alpha(y) \leq \mu_P^\alpha(e)$, $x, y \neq e \in P$.

Proof : If μ is an S-α fuzzy semigroup of G relative to P, then μ_P^α is a fuzzy group. Since $O(P) = p$ which is a prime, all the elements in P are generators of P other than the identity $e \in P$. Let $x \neq e$ and $y \neq e$ be in P.

Since $P = \langle y \rangle, x = y^m$ for some m. Now $\mu_P^\alpha(x) = \mu_P^\alpha(y^m) \geq min\{\mu_P^\alpha(y), \mu_P^\alpha(y^{m-1})\}$ which implies that $\mu_P^\alpha(x) \geq \mu_P^\alpha(y)$, since μ_P^α is a fuzzy group. Similarly, we can prove that $\mu_P^\alpha(y) \geq \mu_P^\alpha(x)$. By theorem 1.8, $\mu_P^\alpha(y) \leq \mu_P^\alpha(e)$. Conversely, assume that $\mu_P^\alpha(x) = \mu_P^\alpha(y) \leq \mu_P^\alpha(e), x, y \neq e \in P$. Let $x, y \in P$. Suppose that $x \neq e$ and $y \neq e$. Then either $xy \neq e$ or $xy = e$. If $xy \neq e$, then $\mu_P^\alpha(xy) = \mu_P^\alpha(x) = \mu_P^\alpha(y)$. Therefore, $\mu_P^\alpha(xy) = min\{\mu_P^\alpha(x), \mu_P^\alpha(y)\}$. If $xy = e$, then $min\{\mu_P^\alpha(x), \mu_P^\alpha(y)\} \leq \mu_P^\alpha(e) = \mu_P^\alpha(xy)$.

Suppose that $x = e$ and $y = e$. Then $xy = e$ which gives the required condition. Suppose that $x \neq e$ and $y = e$. Then $xy = x$. $min\{\mu_P^\alpha(x), \mu_P^\alpha(y)\} = \mu_P^\alpha(x) = \mu_P^\alpha(xy)$. Thus $\mu_P^\alpha(xy) \geq min\{\mu_P^\alpha(x), \mu_P^\alpha(y)\}, x, y \in P$. Now if $x \neq e$, then $x^{-1} \neq e$.

If $x = e$, then $x^{-1} = x$. Therefore in both two cases $\mu_P^\alpha(x^{-1}) = \mu_P^\alpha(x)$. □

Remark 2.13 From theorem 2.8, we conclude that the S-α level subgroups of an S-α fuzzy semigroup A relative to P form a chain. Since $A_P^\alpha(x) \leq A_P^\alpha(e), x \in P$, $A^\alpha_{P_{t_0}}$, where $A_P^\alpha(e) = t_0$, is the smallest S-α level subgroup with respect to P and thus we have the chain $\{e\} = A^\alpha_{P_{t_0}} \subset A^\alpha_{P_{t_1}} \subset A^\alpha_{P_{t_2}}, \ldots\ldots \subset A^\alpha_{P_{t_r}} = P$, where $t_0 > t_1 > t_2 \ldots\ldots > t_r$. Let us denote this chain of S-α level subgroups by $C(A_P^\alpha)$. Since all the subgroups of P do not form a chain, not all the subgroups of P are S-α level subgroups of a given S-α fuzzy semigroup A relative to P. Thus we may be interested to find an S-α fuzzy semigroup A relative to a group P which accommodates as many subgroups of P as possible in $C(A_P^\alpha)$. Such an S-α fuzzy semigroup is constructed in the following theorems.

Theorem 2.14 Let G be an S-semigroup relative to a finite cyclic group P of prime order p. Then there exists an S-α fuzzy semigroup A of G relative to P such that $C(A_P^\alpha)$ is a maximal chain of length 2.

Proof : Let $A : G \rightarrow [0,1]$ be defined as

$$A(x) = \begin{cases} t_0, & if \;\; x = e \\ t_1, & if \;\; x \neq e \in P \end{cases}$$

where $t_0, t_1 \in [0,1]$ and $t_0 > t_1$.

By taking $\alpha \geq t_0$, it can be seen that

$$A_P^\alpha(x) = \begin{cases} t_0, & if \;\; x = e \\ t_1, & if \;\; x \neq e \in P \end{cases}$$

Clearly, $A_P^\alpha(x) = t_1 < t_0 = A_P^\alpha(e), x \neq e \in P$. Thus by theorem 2.12, A is an S-α fuzzy semigroup of G relative to P. Also $A^\alpha_{P_{t_0}} = (e)$ and $A^\alpha_{P_{t_1}} = P$. Therefore $A^\alpha_{P_{t_0}} \subset A^\alpha_{P_{t_1}}$ which is the required chain $C(A_P^\alpha)$ and this is the maximal chain of length 2. □

Theorem 2.15 Let G be an S-semigroup relative to a finite cyclic group P of prime order p such that $P = H_{p_1} H_{p_2} \ldots H_{p_r}$ where H_{p_i}' s are cyclic subgroups of P of prime order p_i. Then there exists an S-α fuzzy semigroup A of G relative to P such that $C(A_P^\alpha)$ is a maximal chain of length $r+1$.

Proof : We shall prove the theorem by mathematical induction on r. If $r = 1$, then $P = H_{p_1}$ which is a cyclic group of prime order $p_1 = p$. By theorem 2.14, there exists an S-α fuzzy semigroup A of G relative to P such that $C(A_P^\alpha)$ is a maximal chain of length 2. Now we take $r > 1$ and assume that the theorem is true for the integers $\leq r-1$. Let $K = H_{p_1}H_{p_2}...H_{p_{r-1}}$. Then $P = KH_{p_r}$. Let us assume the following notations $\hat{H}_{p_1} = H_{p_1} - \{e\}, \widehat{H_{p_1}H_{p_2}} = H_{p_1}H_{p_2} - H_{p_1}...$ $\hat{P} = \widehat{KH_{p_r}} = H_{p_1}H_{p_2}...H_{p_r} - K$. Let $A : G \to [0,1]$ be defined as $A(e) = t_0, A(\widehat{H_{p_1}}) = t_1, A(\widehat{H_{p_1}H_{p_2}}) = t_2 ... A(\hat{K}) = t_{r-1}, A(\widehat{KH_{p_1}}) = t_r$, where $t_0 > t_1 > t_2 ... > t_r$ and $t_i \in [0,1]$. If $\alpha \geq t_0$, then $A_P^\alpha(e) = t_0, A_P^\alpha(\widehat{H_{p_1}}) = t_1, A_P^\alpha(\widehat{H_{p_1}H_{p_2}}) = t_2 ... A_P^\alpha(\widehat{KH_{p_r}}) = t_r$. Now we prove that A_P^α is a fuzzy group. Let $x, y \in P$ Case(i): Suppose that $x, y \in K$. Then $xy \in K$. Since $K = H_{p_1}H_{p_2}...H_{p_{r-1}}$, by mathematical induction, $A_P^\alpha(xy) \geq min\{A_P^\alpha(x), A_P^\alpha(y)\}$ and $A_P^\alpha(x^{-1}) = A_P^\alpha(x)$. Case(ii): Suppose that $x \in K$ and $y \notin K$. Then $xy \notin K$. Otherwise $y \in K$ which is a contradiction. Then $xy \in \widehat{KH_{p_r}}$ and $y \in \widehat{KH_{p_r}}$ which implies that $A_P^\alpha(xy) = t_r$ and $A_P^\alpha(y) = t_r$. Since $x \in K, A_P^\alpha(x) \geq t_{r-1}$. Therefore $A_P^\alpha(xy) \geq min\{A_P^\alpha(x), A_P^\alpha(y)\}$ and $A_P^\alpha(x^{-1}) = A_P^\alpha(x)$. Case(iii): Suppose that $x, y \notin K$. Then $xy \notin K$ or $xy \in K$. If $xy \notin K$, then $A_P^\alpha(xy) = t_r = min\{A_P^\alpha(x), A_P^\alpha(y)\}$. If $xy \in K$, then $A_P^\alpha(xy) \geq t_{r-1}$ which gives $A_P^\alpha(xy) \geq min\{A_P^\alpha(x), A_P^\alpha(y)\}$. Since $x \notin K$, $x^{-1} \in K$ or $x^{-1} \notin K$. Then it can be proved that $A_P^\alpha(x^{-1}) = A_P^\alpha(x)$.

Thus from the above cases, we conclude that A_P^α is a fuzzy group and hence A is an S-α fuzzy semigroup relative to P. By definition of A_P^α, it is easy to see that $A_{P_{t_0}}^\alpha = \{e\}, A_{P_{t_1}}^\alpha = H_{p_1}, A_{P_{t_2}}^\alpha = H_{p_1}H_{p_2} ... A_{P_{t_r}}^\alpha = KH_{p_r}$. Since $\{e\} \subset H_{p_1} \subset H_{p_1}H_{p_2} \subset KH_{p_r}$, $A_{P_{t_0}}^\alpha \subset A_{P_{t_1}}^\alpha \subset A_{P_{t_2}}^\alpha ... \subset A_{P_{t_r}}^\alpha$ which is the required $C(A_P^\alpha)$ and this $C(A_P^\alpha)$ is a maximal chain of length $r+1$. □

Remark 2.16 In a similar way, it may be found an S-α fuzzy semigroup A relative to a group P with maximal chain $C(A_P^\alpha)$ in the following cases

(i) P is a cyclic p-group.
(ii) P is a direct product of cyclic p-subgroups.
(iii) P is a finite abelian group.

Theorem 2.17 Let G be an S-semigroup relative to a cyclic group P of order p^n, where p is a prime and let A be an S-α fuzzy semigroup of G relative to P. Then for $x, y \in P$,

(i) if $O(x) > O(y)$, then $A_P^\alpha(y) \geq A_P^\alpha(x)$

(ii) if $O(x) = O(y)$, then $A_P^\alpha(x) = A_P^\alpha(y)$

Proof : $P \subset G$ is a cyclic group and $O(P) = p^n$, p-prime. We prove the theorem by induction on n. Since A is an S-α fuzzy semigroup of G relative to P, A_P^α is a fuzzy group. For $n = 1$, $O(P) = p$, Let $x, y \in P$. If $x \neq e$ and $y \neq e$, then $O(x) = O(y)$ and by theorem 2.12, $A_P^\alpha(x) = A_P^\alpha(y)$. If $x \neq e$, and $y = e$, then $O(x) = p$ and $O(y) = 1$. Therefore $O(x) \geq O(y)$ and $< x >= P$. Since $y \in P$, $y = x^m$ for some m. Now $A_P^\alpha(y) = A_P^\alpha(x^m) \geq A_P^\alpha(x)$, since A_P^α is a fuzzy group. Thus the theorem is true for $n = 1$. Let $n \geq 1$ and assume that the theorem is true for all integers $\leq n-1$. Now $O(P) = p^n$. Let H be a subgroup of P order p^{n-1}. Let $x, y \in P$. If $x, y \in H$, then by induction, the result is true. Suppose that $x \notin H$ and $y \in H$. Then $O(x) = p^n$ and $O(y) = p^r$, $r \leq n-1$. Therefore $< x >= P$ and $O(x) > O(y)$. Since $y \in P$, as we proved above, $A_P^\alpha(y) \geq A_P^\alpha(x)$. Now suppose that $x, y \notin H$. Then $O(x) = p^n = O(y)$ and hence $< x >= P =< y >$. Therefore $y = x^m$ and $x = y^l$ for some integers l and m. As we proved above, we have $A_P^\alpha(x) \geq A_P^\alpha(y)$ and $A_P^\alpha(y) \geq A_P^\alpha(x)$. Hence the result follows. □

Theorem 2.18 Let G be an S-semigroup relative to a finite cyclic group P and let A be a fuzzy subset of G such that $Im(A_P^\alpha) = \{t_0, t_1, t_2 ..., t_r\}$ where $t_0 > t_1 > t_2 > ... > t_r$ and $\alpha > t_0$. Then A is an S-α fuzzy semigroup of G relative to P if and only if there exists a maximal chain of subgroups of P, $\{e\} = C_0 \subset C_1 \subset C_2 ...... \subset C_r = P$ such that $A\{e\} = t_0, A(\widehat{C_1}) = t_1, ..., A(\widehat{C_r}) = t_r$ where $\widehat{C_i} = C_i - C_{i-1}, i = 1,2,...,r$.

Proof : $P \subset G$ is a finite cyclic group. $Im(A_P^\alpha) = \{t_0, t_1, t_2 ..., t_r\}$ where $t_0 > t_1 > t_2 > ... > t_r$. Let $\alpha > t_0$. Suppose that there exists a maximal chain of subgroups of P, $\{e\} = C_0 \subset C_1 \subset C_2 ... \subset C_r = P$ such that $A\{e\} = t_0, A(\widehat{C_1}) = t_1, ..., A(\widehat{C_r}) = t_r$ where $\widehat{C_i} = C_i - C_{i-1}, i = 1,2,....,r$. Since $\alpha > t_0$, $A_P^\alpha\{e\} = t_0, A_P^\alpha(\widehat{C_1}) = t_1, ..., A_P^\alpha(\widehat{C_r}) = t_r$. We will prove that A_P^α is a fuzzy group. Let $x, y \in P$. Suppose that $x, y \in C_i$ and $x, y \notin C_{i-1}$. Then $x, y \in \widehat{C_i}$ which implies that $A_P^\alpha(x) = t_i$ and $A_P^\alpha(y) = t_i$. Also $xy \in \widehat{C_i}$ or $xy \in \widehat{C_{i-1}}$. This implies that $A_P^\alpha(xy) \geq t_i = min\{t_i, t_i\} = min\{A_P^\alpha(x), A_P^\alpha(y)\}$, since $t_{i-1} > t_i$. Suppose that $x \in C_i$, but $x \notin C_{i-1}$ and $y \in C_j$, but $y \notin C_{j-1}$. Let

$i > j$. Therefore $x \in \widehat{C_i}$ and $y \in \widehat{C_j}$ which gives $A_P^\alpha(x) = t_i$ and $A_P^\alpha(y) = t_j$. Since $i > j, xy \in C_i$ implies $A_P^\alpha(xy) \geq t_i = min\{t_i, t_j\}$ $= min\{A_P^\alpha(x), A_P^\alpha(y)\}$ .Since $x \in C_i, x^{-1} \in C_i$. Therefore $x^{-1} \in \widehat{C_i}$ or $x^{-1} \in C_{i-1}$. If $x^{-1} \in \widehat{C_i}$, then $A_P^\alpha(x^{-1}) = t_i = A_P^\alpha(x)$. If $x^{-1} \in C_{i-1}$, then $A_P^\alpha(x^{-1}) \geq t_i = A_P^\alpha(x)$. Therefore A_P^α is a fuzzy group and hence A is an S-α fuzzy semigroup relative to P.

Conversely, assume that A is an S-α fuzzy semigroup relative to P. Then by corollary 2.9 $A_{P_{t_0}}^\alpha, A_{P_{t_1}}^\alpha, \ldots A_{P_{t_r}}^\alpha$ are the only S-α level subgroups of A relative to P. Also by remark 2.13, these subgroups of P form a chain $C(A_P^\alpha)$, $A_{P_{t_0}}^\alpha \subset A_{P_{t_1}}^\alpha \subset \ldots \subset A_{P_{t_r}}^\alpha$. Here $A_{P_{t_0}}^\alpha = \{e\}$ and $A_{P_{t_r}}^\alpha = P$. If $C(A_P^\alpha)$ is maximal, then by taking $C_i = A_{P_{t_i}}^\alpha$ this chain will satisfy the required condition. Suppose that $C(A_P^\alpha)$ is not maximal. We refine $C(A_P^\alpha)$ by introducing subgroups of P. Let we denote this new chain as $C_0 \subset C_1 \subset C_2 \ldots \subset C_s$, where $C_0 = A_{P_{t_0}}^\alpha = \{e\}$ and $C_s = A_{P_{t_r}}^\alpha = P$. Let $A_{P_{t_1}}^\alpha = C_j$ for some j. Therefore for all C_i between C_0 and C_j, $A_P^\alpha(\widehat{C_i}) = t_1$ and for all C_k between $A_{P_{t_i}}^\alpha$ and $A_{P_{t_{i+1}}}^\alpha$, $A_P^\alpha(\widehat{C_k})$ $= t_{i+1} \ldots A_P^\alpha(\widehat{C_s}) = t_r$. Here $\widehat{C_1} = C_1 - C_0, \widehat{C_2}$ $= C_2 - C_1, \ldots \widehat{C_s} = C_s - C_{s-1}$. Since $\alpha > t_0$, $A\{e\} = t_0, \ldots, A(\widehat{C_r}) = t_r$ and hence the require condition is satisfied. □

Corollary 2.19 If G is an S-semigroup relative to a cyclic group P of order p^r, then the necessary and sufficient condition for a fuzzy subset A of G to be an S-α fuzzy semigroup relative to P is that for all elements x such that $O(x) = p^i$ we have $A_P^\alpha(x) = t_i$, $i = 0,12,\ldots,r$ and $t_0 > t_1 > t_2 \ldots > t_r$.

Proof : By theorem 2.17 and 2.18, proof is obvious. □

References

1. Zadeh L. A., Fuzzy sets, *Information and Control, 8,* 338-353 (1965).
2. Rosenfeld A., Fuzzy groups, *Journal of Mathematical analysis and application, 35,* 512-517 (1971).
3. Sivaramakrishna Das P., Fuzzy Groups and Level Subgroups, *Journal of Matematical Analysis and Applications, 84,* 264-269 (1981).
4. Rajeshkumar, Fuzzy Algebra, University of Delhi Publication Division (1993).
5. Vasantha Kandhasamy W. B., Smarandache Fuzzy Algebra, American Research Press, Rehoboth (2003).
6. Sharma P.K., α-Fuzzy subgroups, International *Journal of fuzzy Mathematics and systems, 3(1),* 47-59 (2013).
7. Gowri R., Rajeswari T., S-α fuzzy semigroups, *International Journal of Mathematical Sciences and Engineering Applications, 9(1),* 307-318 (2015).
8. Gowri. R and Rajeswari. T, "S-α anti fuzzy semigroups", *International Journal of Engineering Science and Innovative Technology,* Volume *4,* Issue *6,* 108-118 (2015).
9. Gowri. R. and Rajeswari. T., "Some properties of S-α anti fuzzy semigroups", *International Journal Mathematical Trends and Technology,* Vol *35(1),* 24-31 (2016).