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Ricci Solitons On Quasi-Sasakian Manifold

SUSHIL SHUKLA and SHIKHA TIWARI

Department of Applied Science (Mathematics)
Madan Mohan Malviya University of Technology, Gorakhpur
Corresponding Author Email:- sushilcws@gmail.com
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Abstract

The object of present paper is to study a special type of metric called $*$-Ricci soliton on Quasi-Sasakian manifold.

Key words: Ricci soliton, Quasi-Sasakian manifold, *Einstein manifold.*

Ams Subject Classification (2010):53C15,53C25

1. **Introduction**

Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field, and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0 \tag{1.1}$$

where S is a Ricci tensor of M and s denotes the Lie derivative operator along the vector field V.The Ricci soliton is said to be shrinking, steady, and expanding accordingly as λ is negative, zero, and positive, respectively[1]. In 1967, D. E. Blair[2] introduced the notion of quasi-Sasakian manifold to unify Sasakian and cosympletic manifolds and in 1977, S. Kanemaki[18] studied quasi-Sasakian manifolds. The authors in [3–7] have studied Ricci solitons in contact and Lorentzian manifolds. G. Kaimakamis and K. Panagiotidou[8] initiated the notion of $*$-Ricci soliton where they essentially modified the definition of Ricci soliton by replacing the Ricci tensor Ric in (1.1) with the $*$-Ricci tensor Ric^*. A pseudo-Riemannian metric g on a smooth manifold M is called a $*$-Ricci soliton if there exists a smooth vector field V, such that

where

$$2(\pounds_V g)(X,Y) + Ric^*(X,Y) = \lambda g(X,Y), \tag{1.2}$$

$$Ric^*(X,Y) = \frac{1}{2}(trace\{\varphi, R(X,\varphi Y)\}) \tag{1.3}$$

for all vector fields X, Y on M.

The notion of $*$-Ricci tensor was first introduced by S. Tachibana[9] on almost Hermitian manifolds and further studied by T. Hamada[10] on real hypersurfaces of non-flat complex space forms.

In the present paper, we have studied $*$-Ricci soliton on Quasi-Sasakian manifold and prove the following result:

Theorem: Let M (φ, ξ, η, g) *be a* $(2n + 1)$*-dimensional Quasi-Sasakian manifold. If g is a* $*$*-Ricci soliton on M, then either M is* D*-homothetic to an Einstein manifold, or the Ricci tensor of M with respect to canonical paracontact connection vanishes. In the first case, the soliton vector field is Killing and in the second case, the soliton vector field leaves* φ *invariant.*

2. Preliminaries :

Let M be an almost contact manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a compatible Riemannian metric g satisfying

$$\phi = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0 \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), g(X, \xi) = \eta(X)$$

for all $X, Y \in \chi(M)$.

An almost contact metric structure is called quasi-Sasakian if it is normal and its fundamental form is closed, that is, for every $X, Y \in T\bar{M}$.

$$[\phi,\phi](X,Y) + d\eta(X,Y)\xi = 0,$$

$$d\varphi = 0, \ \varphi(X,Y) = g(X, \phi Y).$$

This was first introduced by Blair[2]. An almost contact metric manifold M is a quasi-Sasakian manifold if and only if [19]

$$(\nabla_X \varphi)(Y) = \beta(g(X,Y)\xi - \eta(Y)X), X, Y \in T(M) \tag{2.3}$$

Where ∇ is Levi-Civita connection of the Riemannian metric g.

From the above equation it follows that

$$\nabla_X \xi = -\beta\phi(X), \quad XT(M) \tag{2.4}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y) = -\beta g(\phi X, Y) \tag{2.5}$$

Moreover, the curvature tensor R and Ricci tensor S satisfy[17]

$$R(X,Y)\xi = -(X\beta)\varphi Y + (Y\beta)\varphi X + \beta^2\{\eta(Y)X - \eta(X)Y\} \tag{2.6}$$

Let M be a three-dimensional quasi-Sasakian manifold. The Ricci tensor S of M is given by[20]

$$S(Y, Z) = (r/2 - \beta^2) g(Y, Z) + (3\beta^2 - r/2)\eta(Y)\eta(Z) - \eta(Y)d\beta(\varphi Z) - \eta(Z)d\beta(\varphi Y) \quad (2.7)$$

Which yields

$$QX = (r/2 - \beta^2)X + (3\beta^2 - r/2)\eta(X)\beta - \eta(X)(\varphi \operatorname{grad} \beta) - d\beta(\varphi X)\xi, \quad (2.8)$$

where the gradient of a function f is related to the exterior derivative df by the formula $df(X) = g(grad\, f, X)$, r is the scalar curvature of M and S denotes the Ricci tensor defined by $S(X, Y) = g(QX, Y)$, where Q is the Ricci operator.

Lemma 1. Let $M(\varphi, \xi, \eta, g)$ be a Quasi-Sasakian manifold. Then
(i) $\nabla_\xi Q = 0$, *and (ii)* $(\nabla_X Q)\xi = Q\varphi X + \lambda\varphi X.$

Proof: Since ξ is Killing, we have $\pounds_V\, Ric = 0$. This implies $(\pounds_\xi Q)X = 0$ for any vector field X on M. From which it follows that

$$0 = \pounds_\xi(QX) - Q(\pounds_\xi X)$$
$$= \nabla_\xi QX + \nabla_{QX}\xi - Q(\nabla_\xi X) + Q(\nabla_X \xi)$$
$$= (\nabla_\xi Q)X + \nabla_{QX}\xi + Q(\nabla_X \xi).$$

Using (2.4) in the above equation gives $\nabla_\xi Q = Q\varphi - \varphi Q$. Since the Ricci operator Q commutes with φ on Quasi-Sasakian manifold, we have (*i*). Next, taking covariant differentiation of (2.8) along an arbitrary vector field X on M and using (2.4), we obtain (*ii*). This completes the proof.
If the Ricci tensor of a Quasi-Sasakian manifold M is of the form
$Ric(X, Y) = Ag(X, Y) + B\eta(X)\,\eta(Y),$
for any vector fields X, Y on M, where A and B being constants, then M is called an η-Einstein manifold.
The 1-form η is determined up to a horizontal distribution and hence D = Kerη *are* connected by $\tilde{\eta} = \sigma\eta$ *for* a positive smooth function

σ *on* a paracontact manifold M. This paracontact form $\bar{\eta}$ defines the structure tensor $(\bar{\varphi}, \bar{\xi}, \bar{g})$ corresponding to η using the condition given in the paper[11]. We call the transformation of the structure tensors given by Lemma 4.1 of [11] a gauge (conformal) transformation of paracontact pseudo-Riemannian structure. When σ is constant this is a D-homothetic transformation. Let $M(\varphi, \xi, \eta, g)$ be a paracontact manifold and

$$\bar{\phi} = \phi, \bar{\xi} = \frac{1}{\alpha}\xi, \bar{\eta} = \alpha\eta, \bar{g} = \alpha g + (\alpha^2 - \alpha)\eta \otimes \eta \;\; \alpha = const. \neq 0$$

to be D-homothetic transformation. Then $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also a para contact structure. Using the formula appeared in[11] for D-homothetic deformation, one can easily verify that if $M(\varphi, \xi, \eta, g)$ is a (2n+1) - dimensional ($n > 1$) η-Einstein Quasi-Sasakian structure with scalar curvature $r \neq 2n$, then there exists a constant *á* such that $M(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is an Einstein Quasi-Sasakian structure. So we have following result.

Lemma 2. Any $(2n+1)$*-dimensional* η*-Einstein Quasi-Sasakian manifold with scalar curvature not equal to* $2n$ *is* D*-homothetic to an Einstein manifold.*

3. Proof of Theorem :

First, we state and prove some lemmas which will be used to prove Theorem.

Lemma 3. The $*$-Ricci tensor on a $(2n+1)$*-dimensional Quasi-Sasakian manifold* $M(\varphi, \xi, \eta, g)$ *is given by*

$$Ric^*(X,Y) = -Ric(X,Y) - (2n-1)g(X,Y) - \eta(X)\eta(Y) \tag{3.1}$$

for any vector fields X, Y on M.

Proof: The Ricci tensor Ric of a $(2n+1)$-dimensional Quasi-Sasakian manifold $M(\varphi, \xi, \eta, g)$ satisfies the relation (c.f. Lemma 3.15 in [11]):

$$Ric(X,Y) = \sum_{i=1}^{2n+1} R'(X, \phi Y, e_i, \phi e_i) - (2n-1)g(X,Y) - \eta(X)\eta(Y) \tag{3.2}$$

for any vector fields X, Y on M. By the skew-symmetric property of φ, we have

$$\sum_{i=1}^{2n+1} R'(X, \phi Y, e_i, \phi e_i) = \sum_{i=1}^{2n+1} R(X, \phi Y, e_i, \phi e_i) = \sum_{i=1}^{2n+1} g(\phi R(X, \phi Y), e_i, e_i)$$

By this, (3.2) becomes

$$\sum_{i=1}^{2n+1} g(\phi R(X, \phi Y), e_i, e_i) = -2Ric(X,Y) - 2(2n-1)g(X,Y) - 2\eta(X)\eta(Y) \tag{3.3}$$

By (1.3) and (3.3), we have (3.1).

Lemma 4. For a Quasi-Sasakian manifold, we have the following relation

$$(\pounds_V \eta)(\xi) = -\eta(\pounds_V \xi) = \lambda. \tag{3.4}$$

Proof: By virtue of Lemma 3, the $*$-Ricci soliton equation (1.2) can be expressed as

$$(\pounds_V g)(X,Y) = 2Ric(X,Y) + 2(2n-1+\lambda)g(X,Y) + 2\eta(X)\eta(Y). \tag{3.5}$$

Taking $Y = \xi$ in (3.5) and using (2.7) we have $(\pounds_V g)(X, \xi) = 2\lambda\eta(X)$. Lie-differentiating the equation $\eta(X) = g(X, \eta)$ along V and by (3.5), we have

$$(\pounds_V \eta)(X) - g(\pounds_V \xi, X) - 2\lambda\eta(X) = 0. \tag{3.6}$$

Now, Lie-derivative of $g(\xi, \xi) = 1$ along V and equation (3.6) completes proof.

Lemma 5. *Let $M(\varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional Quasi-Sasakian manifold. If g is a $*$-Ricci soliton, then M is an η-Einstein manifold and the Ricci tensor can be written as*

$$Ric(X,Y) = -\left[2n-1+\frac{\lambda}{2}\right]g(X,Y) + \left[\frac{\lambda}{2}-1\right]\eta(X)\eta(Y) \tag{3.7}$$

for any vector fields X, Y on M.

Proof: Taking covariant differentiation of (3.5) along an arbitrary vector field Z, we get

$$(\nabla_Z \pounds_V g)(X,Y) = 2\{(\nabla_Z Ric)(X,Y) - g(X, \varphi Z)\eta(Y) - g(Y, \varphi Z)\eta(X)\}. \tag{3.8}$$

According to Yano[12], we have

$$(\pounds_V \nabla_Z g - \nabla_Z \pounds_V g - \nabla_{[V,Z]g})(X,Y) = -g((\pounds_V \nabla)(Z,X), Y) - g((\pounds_V \nabla)(Z,Y), X),$$

for any vector fields X, Y, Z on M.

In view of the parallelism of the pseudo-Riemannian metric g, we have from above relation

$$(\nabla_Z £_V g)(X, Y) = g((£_V \nabla)(Z, X), Y) + g((£_V \nabla)(Z, Y), X). \tag{3.9}$$

From (3.8) and (3.9), we have

$$g((£_V \nabla)(Z, X), Y) + g((£_V \nabla)(Z, Y), X)$$
$$= 2\{(\nabla_Z Ric)(X, Y) - g(X, \varphi Z)\eta(Y) - g(Y, \varphi Z)\eta(X)\}. \tag{3.10}$$

Which gives

$$g((£_V \nabla)(X, Y), Z) = -(\nabla_Z Ric)(X, Y) + (\nabla_X Ric)(Y, Z)$$
$$+ (\nabla_Y Ric)(Z, X) + 2g(X, \varphi Z)\eta(Y)$$
$$+ 2g(Y, \varphi Z)\eta(X). \tag{3.11}$$

Taking ξ in place of Y in (3.11) and Lemma 1, we get

$$(£_V \nabla)(X, Y) = 2(2n-1)\varphi X + 2Q\varphi X. \tag{3.12}$$

Differentiating (3.12) covariantly along an arbitrary vector field Y on M and using the relations (2.3) and (2.8), we have

$$(\nabla_Y £_V \nabla)(X, \xi) + (£_V \nabla)(X, \varphi Y)$$
$$= 2\{(\nabla_Y Q)\varphi X + \eta(X)QY + (2n-1)\eta(X)Y + g(X, Y)\xi\}. \tag{3.13}$$

According to Yano[12] we have

$$(£_V R)(X, Y)Z = (\nabla_X £_V \nabla)(Y, Z) - (\nabla_Y £_V \nabla)(X, Z). \tag{3.14}$$

Taking ξ in place of Z in (3.14) and by (3.13), we have

$$(£_V R)(X, Y)\xi + (£_V \nabla)(Y, \varphi X) - (£_V \nabla)(X, \varphi Y)$$
$$= 2\{(\nabla_X Q)\varphi Y - (\nabla_Y Q)\varphi X + \eta(Y)QX - \eta(X)QY$$
$$+ (2n-1)(\eta(Y)X - \eta(X)Y)\}. \tag{3.15}$$

Taking ξ for Y in (3.15), then using (2.8), (3.12) and Lemma 1, we have $(£_V R)(X, \xi)\xi = 4\{QX + (2n-1)X + \eta(X)\xi\}$. (3.16)

Taking Lie-derivative of (2.6) along V and by (2.5) and (3.4) we have

$$(£_V R)(X, \xi)\xi = (£_V \eta)(X)\xi - g(£_V X, \xi) - 2\lambda X. \tag{3.17}$$

Comparing (3.16) with (3.17), and use of (3.6), gives the required result.

Proof of Theorem: By (3.7), the soliton equation (3.5) can be written as

$$(£_V g)(X, Y) = \lambda\{g(X, Y) + \eta(X)\eta(Y)\}. \tag{3.18}$$

Taking Lie-differentiation of (3.7) along the vector field V and using (3.5) we have

$$(£_V Ric)(X,Y) = \left(\frac{\lambda}{2} - 1\right)\{\eta(Y)(£_V \eta)(X) + \eta(X)(£_V \eta)(Y)\}$$
$$- \left[\frac{\lambda}{2} + 2n - 1\right]\lambda\{g(X,Y) + \eta(X)\eta(Y)\} \tag{3.19}$$

Differentiating (3.7) covariantly along an arbitrary vector field Z on M and using (2.4) we have

$$(\nabla_Z Ric)(X,Y) = \left(1 - \frac{\lambda}{2}\right)\{g(X, \phi Z)\eta(Y) + g(Y, \phi Z)\eta(X)\} \tag{3.20}$$

By (3.20), equation (3.11) becomes

$$(\pounds_V \nabla)(X, Y) = -\lambda \{\eta(Y)\varphi X + \eta(X)\varphi Y\}. \tag{3.21}$$

Differentiating (3.21) covariantly along an arbitrary vector field Z on M and by (2.3) and (2.4), we have

$$(\nabla_Z \pounds_V \nabla)(X,Y) = \lambda\{g(Y, \varphi Z)\varphi X + g(X, \varphi Z)\varphi Y + g(X, Z)\eta(Y)\xi + g(Y,Z)\,\eta(X)\,\xi - 2\eta(X)\,\eta(Y)\,Z\}. \tag{3.22}$$

Using (3.22) in (3.14) and using (2.4) we have

$$(\pounds_V R)(X, Y)Z = \lambda\{g(\varphi X, Z)\varphi Y - g(\varphi Y, Z)\varphi X + 2g(\varphi X, Y)\varphi Z + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi - 2\eta(Y)\,\eta(Z)X + 2\eta(X)\eta(Z)Y\}. \tag{3.23}$$

Contracting (3.23) over Z, we get

$$(\pounds_V Ric)(Y, Z) = 2\lambda\{g(Y, Z) - (2n+1)\eta(Y)\eta(Z)\}. \tag{3.24}$$

By (3.19) and (3.24), we have

$$\left(\frac{\lambda}{2}-1\right)\{\eta(Y)(\pounds_V\eta)(Z)+\eta(Z)(\pounds_V\eta)(Y)\} - \left[\frac{\lambda}{2}+2n-1\right]\lambda\{g(Y,Z)+\eta(Y)\eta(Z)\} = 2\lambda\{g(Y,Z)-(2n+1)\eta(Y)\eta(Z)\} \tag{3.25}$$

Replacing Y by $\varphi^2 Y$ in (3.25) and then using (2.1) and (3.4) we get

$$\left(\frac{\lambda}{2}-1\right)\left\{(\pounds_V\eta)(Y)\eta(Z) = \lambda\left[1+2n+\frac{\lambda}{2}\right]g(Y,Z) - 2n\lambda\eta(Y)\eta(Z)\right\} \tag{3.26}$$

By (3.26) and (3.25) and then replacing Z by φZ, we have

$$\lambda\left[1+2n+\frac{\lambda}{2}\right]g(Y,\phi Z) = 0 \tag{3.27}$$

As $\varphi(Y, Z) = g(Y, \varphi Z)$ is non-vanishing everywhere on M, so either $\lambda = 0$ or $\lambda = -2\,(2n + 1)$.

Case I: If $\lambda = 0$, from (3.18) we have $\pounds_V g = 0$, therefore, V is Killing. From (3.7) we have

$$Ric(X, Y) = -(2n-1)g(X, Y) - \eta(X)\eta(Y). \tag{3.28}$$

Contracting the equation (3.28) we have $r = -4n^2$, where r is the scalar curvature of the manifold M. This shows that M is a η-Einstein manifold with scalar curvature $r \neq 2n$. So, M is D- homothetic to an Einstein manifold.

Case II: If $\lambda = -2\,(2n + 1)$, then taking ξ in place of Z in (3.26) and then replace Y by φY, the resulting equation gives

$$(\frac{\lambda}{2}\ \ 1)(\pounds_V\,\eta)(\Phi Y) = 0.$$

Since $\lambda = -2(2n + 1)$, we have $\lambda \neq 2$. Thus we have $(\pounds_V\,\eta)\,(\varphi Y) = 0$.
Replacing Y by φY and using (2.1), we have

$$(\pounds_V\,\eta)(Y) = -2\,(2n+1)\,\eta(X). \tag{3.29}$$

Taking exterior differentiation d on (3.29) we have

$(£_V d\eta)(X, Y) = -2(2n+1)g(X, \varphi Y),$ (3.30)

as d commutes with $£_V$.

Taking the Lie-derivative of $d\eta(X, Y) = g(X, \varphi Y)$ along the soliton vector field V provides

$(£_V d\eta)(X, Y) = (£_V g)(X, \varphi Y) + g(X, (£_V \varphi)Y).$ (3.31)

From (3.18) we have

$(£_V g)(X, \varphi Y) = -2(2n+1)g(X, \varphi).$ (3.32)

Using (3.30) and (3.32) in (3.31) we have $£_V \varphi = 0$. Therefore, soliton vector field V leaves φ invariant.

Putting $\lambda = -2(2n+1)$ in (3.7) we have

$Ric(X, Y) = 2g(X, Y) - (2n+2)\eta(X)\eta(Y).$ (3.33)

Contracting (3.33) we obtain $r = 2n$ (i.e., the manifold M cannot be D- homothetic to an Einstein manifold.

Ricci tensor $\widetilde{Ric}$ *of a (2n+1) dimensional Quasi-Sasakian manifold with respect to canonical paracontact connection* $\tilde{\nabla}$ *is defined as [11]*

$$\widetilde{Ric}(X, Y) = Ric(X,Y) - 2g(X,Y) + (2n+2)\eta(X)\eta(Y) \quad (3.34)$$

Using (3.33) in (3.34) we have $\widetilde{Ric}(X, Y) = 0$. Therefore, the Ricci tensor with respect to the connection $\tilde{\nabla}$ vanishes. This completes the proof of theorem.

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