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## Section A



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## Relation Meet and Relation Join Matrices on Posets

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### Abstract

We define relation meet and relation join matrices on two different posets R and S of a lattice with respect to a complex-valued function f on P by  $(X, Y)_f = (f(x_i \wedge y_i))$  and  $[X, Y]_f = (f[x_i \vee y_i])$  respectively. Also we present some examples for the determinant and inverse of  $(X, Y)_f$  and  $[X, Y]_f$

*Key words:* Cartesian product, relations and its properties, posets, relation meet matrix, relation join matrix.

**Mathematics subject classification:** 15A57, 11A57, 11C57, 06A57

### 1. Introduction

Let  $S = \{x_1, x_2, x_3, \dots, x_n\}$  be a set of distinct Cartesian product and let f be an arithmetical function. Let  $(S)_f$  denote the  $n \times n$  matrix  $(f(x_i, x_j))$ , the image of the greatest common divisor of  $x_i$  and  $x_j$  as its  $ij$  entry. Analogously, let  $[S]_f$  denote the  $n \times n$  matrix having  $(f[x_i, x_j])$ , the image of the least common multiple of  $x_i$  and  $x_j$  as its  $ij$  entry. That is  $(S)_f = (f(x_i, x_j))$ , and  $[S]_f = (f[x_i, x_j])$ . The matrices  $(S)_f$  and  $[S]_f$  are referred to as the GCD and LCM matrices on S associated with f respectively.

In 1875, Smith<sup>7</sup> calculated  $\det(S)_f$  and  $\det[S]_f$ . Since then a large number of results on GCD and LCM matrices have been presented. Haukkanen generalised the concept of a GCD matrix into a meet matrix and later Korkee and Haukkanen<sup>3,4,5</sup> did the same with the concepts of LCM and join matrices.

Let  $(P, \leq)$  be a locally finite lattice, let  $S = \{x_1, x_2, x_3, \dots, x_n\}$  be a subset of P and let f be a complex-valued function on P. The  $n \times n$  matrix  $(S)_f = (f(x_i \wedge x_j))$  is called the meet matrix on S associated with f and the  $n \times n$  matrix  $[S]_f = (f[x_i, x_j])$  is called the join matrix on S associated with f.

The properties of meet and join matrices have been studied by many authors. Haukkanen<sup>2</sup> calculated  $\det(S)_f$  on an arbitrary set S. Most recently, Altinisik, Tuglu and Haukkanen<sup>1</sup> generalised the concepts of meet and join matrices and defined meet and join matrices on two sets. We present these same definitions.

Let  $R_1 = \{x_1, x_2, x_3, \dots, x_n\}$  and  $R_2 = \{y_1, y_2, y_3, \dots, y_n\}$  be two subsets of P. We define the meet matrix on  $R_1$  and  $R_2$  with respect to f as  $(R_1, R_2)_f = (f(X_i \wedge Y_j))$ . In particular, when  $S = R_1 = R_2 = \{x_1, x_2, x_3, \dots, x_n\}$ . We have  $(S, S)_f$

$= (S)_f$ . Analogously, we define the join matrix on  $R_1$  and  $R_2$  with respect to  $f$  as  $[R_1, R_2] = (f[X_i, Y_i])$ . In particular  $[S, S]_f = [S]_f$ .

2. Basic Definitions :

In This Section, We Define Some preliminary concepts that are needed to understand the summaries of the articles.

Definition : 2.1 :

A **Cartesian product** is a set of all ordered 2-tuples where each “part” is from a given set, denoted by  $A \times B$  and uses parenthesis.

Let  $A = \{a_1, a_2, a_3, \dots, a_m\}$  and  $B = \{b_1, b_2, b_3, \dots, b_n\}$

The Cartesian product  $A \times B$  is defined by a set of pairs  $\{(a_1, b_1), (a_1, b_2), \dots, (a_1, b_n), \dots, (a_m, b_n)\}$

Example : 2.1 Let  $A = \{1, 2, 3\}$   $B = \{a, b, c\}$

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$$

If  $|A| = m, |B| = n$ , then How many pairs in  $A \times B$  ?

Answer :  $mn$

Definition : 2.2 :

Let  $A$  and  $B$  be two sets. A **Binaryrelation** from  $A$  to  $B$  is a subset of a Cartesian product  $A \times B$ . Hence, there are  $2^{mn}$  different relations if  $|A|=m, |B|=n$ .

Example: 2.2 :

Let  $S$  be the set of all students and  $G$  be the set of all grades

$$S = \{Riya, Sam, Sri\} \text{ and } G = \{A, B, C\}$$

The set of all Cartesian products will be

$$S \times G = \{(Riya, A), (Riya, B), (Riya, C), (Sam, A), (Sam, B), (Sam, C), (Sri, A), (Sri, B), (Sri, C)\}$$

The final grade will be a subset of this  $\{(Riya, A), (Sam, B), (Sri, C)\}$

Such a subset of a Cartesian product is called a **Relation**.

Example : 2.3 :

Let  $A$  be the students in the Maths major and  $B$  be the courses the department offers  $A = \{Sri, Sai, Ram\}$  and  $B = \{M201, M202, M301\}$

Then the relation  $R$  as the set that lists all students  $a \in A$  enrolled in class  $b \in B$

$$R = \{(Sri, M201), (Sam, M202), (Sam, M301), (Ram, M301)\}$$

A Relation is when there are multiple mappings between the domain and co-domain.

A Relation on the set  $A$  is a relation from  $A$  to  $A$ . i.e.) the domain and the co-domain are the same.

2.2.1. Representing Binary Relations:

Relations can be represented by a table showing the ordered pair of  $R$ .

Example: 2.4 :

$$A = \{0, 1, 2\} \quad B = \{u, v\} \quad R = \{(0, u), (0, v), (1, v), (2, u)\}$$

$$R = \begin{matrix} & \begin{matrix} u & v \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \end{matrix}$$

2.2.1.1. Matrix Review :

We will only be dealing with zero-one matrices. Each element in the matrix is either 0 or 1.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

These matrices will be used for Boolean operations.

1 is true, 0 is false.

### 2.2.1.2. Relations using Matrices :

List the elements of Sets A and B in a particular order.

Consider a matrix  $M_R = [m_{ij}]$

$$[m_{ij}] = 1, \text{ if } (a, b) \in R$$

$$[m_{ij}] = 0, \text{ if } (a, b) \notin R$$

We actually represent R as a Boolean function over  $A \times B$ .

$R(A, B) = \text{True}$  iff  $(a, b)$  is in R.

We will generally consider relations on a single set.

i.e.) A and B are the same set and the matrix is square.

It is good for How computers view Relations as a2-dimensional array.

### 2.2.2. Properties of Relations:

#### Property 2.1:

A relation R is Reflexive iff MR (matrix of R) has 1 in every position on its main diagonal. In other words, a relation is reflexive if every element is related to itself or  $(a, a) \in R$

Example 2.5:  $R_{\text{div}} = \{(a, b) / a \mid b\}$  on  $A = \{1, 2, 3\}$

$$R_{\text{div}} = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\}$$

$$MR_{\text{div}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

#### Property: 2.2 :

A relation R is Irreflexive iff MR (matrix of R) has 0 in every position on its main diagonal. i.e.) every element is not related to itself or  $(a, a) \notin R$

Example: 2.6 :  $R_{\neq} = \{(a, b) / a \neq b\}$  on  $A = \{1, 2, 3\}$

$$R_{\neq} = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$$

$$MR_{\neq} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

#### Property : 2.3 :

A relation R is Symmetric iff  $m_{ij} = m_{ji}$  i.e.) for every  $(a, b) \in R \Rightarrow (b, a) \in R$

Example : 2.7  $R_{=} = \{(a, b) / a = b\}$

$$R_{=} = \{(1, 1), (2, 2), (3, 3)\}$$

$$MR_{=} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

#### Property : 2.4 :

A relation R is said to be Asymmetry if for every  $(a, b) \in R \Rightarrow (b, a) \notin R$

Example: 2.8  $R_{<} = \{(a, b) / a < b\}$

$$R_{<} = \{(1, 2), (1, 3), (2, 3)\}$$

$$MR_{<} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Note : 2.1 for every value and value in its transposition, if they are not both 1 then it is asymmetry. It must be irreflexive and main diagonal = 0's.

*Property : 2.5 :*

A relation R on a set A is called Anti-symmetry if  $(a,b) \in R$  and  $(b,a) \in R \Rightarrow a = b$  where  $a,b \in R$ .

A relation R is anti-symmetry iff  $m_{ij}=1 \rightarrow m_{ji} = 0 \forall i$  and  $j$ .

*Example: 2.9 :*  $R_{>} = \{(a, b) / a > b\}$   
 $R_{>} = \{(2,1),(3,1),(3,2)\}$

$$MR_{>} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

*Note 2.2 :*

A relation can be neither symmetry nor anti-symmetry.

Consider  $R = \{(a, b) / a = |b|\}$

This is not symmetry,  $\because (4,4) \in R$  but  $(-4,4) \in R$

This is not asymmetry,  $\because (4,4) \in R$  but  $(4,4) \notin R$

This is anti-symmetry because  $(a,b) \in R, (b,a) \in R \Rightarrow a = b$ .

*Property : 2.6 :*

A relation R on a set A is called Transitive if  $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R, \forall a, b, c \in A$ .

*Example: 2.10:*  $R_{\geq} = \{(a,b) / a \geq b\}$   
 $R_{\geq} = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$

$$MR_{\geq} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

i.e.)  $m_{ik} = 1$  if  $a_{ij} = 1$  and  $b_{jk} = 1$

*Definition : 2.3 :*

A relation which satisfies the properties reflexive, anti-symmetry and transitive are said to be partially ordered relation. A set which posses partially ordered relation is called partially ordered set or **poset**.

*Definition : 2.4 :*

The meet denoted by  $\wedge$  of two m-by-n matrices  $(a_{ij})$  and  $(b_{ij})$  of 0's and 1's is an m-by-n matrix  $(m_{ij})$  where  $m_{ij} = a_{ij} \wedge b_{ij}$

$m_{ij}$  = pair wise or conjunction

A meet of two relation matrices perform a Boolean AND on each relative entry of the matrices.

*Example : 2.11* Let  $R_{<} = \{(a,b) / a < b\}$  and  $R_{\leq} = \{(a,b) / a \leq b\}$

$R_{<} = \{(1,2),(1,3),(2,3)\}$  ,  $R_{\leq} = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$

$$\text{Then } MR_{<} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad MR_{\leq} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(MR_{<}) \wedge (MR_{\leq}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Definition : 2.5 :*

The join denoted by  $\vee$  of two m-by-n matrices  $(a_{ij})$  and  $(b_{ij})$  of 0's and 1's is an m-by-n matrix  $(m_{ij})$  where  $m_{ij} = a_{ij} \vee b_{ij}$

$m_{ij}$  = pair wise or disjunction

A join of two relation matrices perform a Boolean OR on each relative entry of the matrices.

*Example 2.12 :* Let  $R_{<} = \{(a,b) / a < b\}$  and  $R_{>} = \{(a,b) / a > b\}$   
 $R_{<} = \{(1,2),(1,3),(2,3)\}$  ,  $R_{>} = \{(2,1),(3,1),(3,2)\}$

$$\text{Then } MR_{<} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad MR_{>} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$(MR_{<}) \vee (MR_{>}) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

*Definition : 2.6 :*

A Boolean product of two relation matrices is similar to matrix multiplication

$$C_{11} = a_{11} * b_{11} + a_{12} * b_{21} + a_{13} * b_{31}$$

Instead of sum of products, it is the conjunction of disjunction

$$C_{11} = (a_{11} b_{11}) \wedge (a_{12} b_{21}) \wedge (a_{13} b_{31})$$

*3. Some Basic Theorems :*

*Theorem 3.1:*

Let R be a Binary relation on A and let M be its relation matrix. Then,

- (i) R is reflexive iff  $m_{ii} = 1, \forall i$
- (ii) R is symmetry iff  $m_{ij} = m_{ji}, \forall i, j$
- (iii) R is anti-symmetric iff  $m_{ij} = 1, \forall i.$

*Proof :*

The result is so obvious.

*Theorem 3.2:*

The number of reflexive relations on a set A where  $|A| = n$  is  $2^{n(n-1)}$ .

*Proof:*

A reflexive relation R on a set A must contain all pairs (a,a) where  $a \in A$ .

All other pairs in R are of the form (a,b) ,  $a \neq b$  such that  $a, b \in A$

$n(n-1)$  pairs are there.

$\therefore$  There are  $2^{n(n-1)}$  subsets on  $n(n-1)$  elements.

*Theorem 3.3:*

The number of possible symmetric relations on a set A where  $|A| = n$  is  $2^{\frac{n^2+n}{2}}$ .

*Proof:*

Consider the matrix representing symmetric relation R on a set with n elements.

The Centre diagonal can have any values. Once, the upper triangle is determined, the lower triangle must be transposed version of the upper one.

How many ways are there to fill in the centre diagonal and upper triangle ?

$$\begin{bmatrix} 1 & . & . & . & 0 & 1 \\ . & 1 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & . \\ 1 & 0 & . & . & . & 1 \end{bmatrix}$$

There are  $n^2$  elements in the matrix. There are  $n$  elements in the centre diagonal. Thus there are  $2^n$  ways to fill in 0's and 1's in the diagonal.

Thus there are  $\frac{n^2 - n}{2}$  elements in centre triangle.

Hence, there are  $2^n * 2^{\frac{n^2 - n}{2}} = 2^{\frac{n^2 + n}{2}}$  possible symmetric relations on a set with  $n$  elements.

*Theorem 3.4:*

The relative  $R^n$  on a set  $A$  is transitive iff  $R^n \subseteq R$ , for  $n = 1, 2, 3, \dots$

*Proof:*

Suppose  $R^n \subseteq R$ , for  $n = 1, 2, 3, \dots$

Let  $(a, b) \in R$  and  $(b, c) \in R$

$\Rightarrow (a, c) \in R \circ R = R$ , by definition of  $R \circ R$

$\Rightarrow (a, c) \in R^2 \subseteq R$

$\Rightarrow (a, c) \in R$

$\Rightarrow R$  is transitive.

Conversely, Suppose  $R$  is transitive.

Let  $P(n) : R^n \subseteq R$ , Mathematical induction

Step 1 :  $P(1)$  says  $R^1 = R$  so,  $R^1 \subseteq R$  is true.

Inductive Step : To show  $P(n) \rightarrow P(n+1)$

i.e.) We have to show if  $R^n \subseteq R$  then  $R^{n+1} \subseteq R$ .

Let  $(a, b) \in R^{n+1}$

Then by the definition of  $R^{n+1} = R^n \circ R$ , there is an element  $x \in A$  so that  $(a, x) \in R$  and  $(x, b) \in R^n \subseteq R$  (inductive hypothesis).

In addition to  $(a, x) \in R$  and  $(x, b) \in R$ ,  $R$  is transitive so  $(a, b) \in R$

$\Rightarrow R^{n+1} \subseteq R$ .

*4. Relation Meet Matrices and Relation join Matrices on Posets :*

*Illustration 4.1:*

Let us try to understand the relation meet matrix through an example.

Let  $R = \{(1,1), (1,2), (1,3), (2,3), (3,1)\}$  and

$S = \{(1,1), (2,2), (3,3)\}$  be two relations on  $A = \{1, 2, 3\}$ .

Find  $M(R \wedge S)$  and  $M(R \vee S)$

*Solution :*

$$\text{Let } MR = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad MS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} M(R \wedge S) &= (MR) \wedge (MS) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$M(R \vee S) = (MR) \vee (MS) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

*Theorem 4.1:*

Let R and S be two binary relations on an arbitrary set A. Then

(i)  $M(R \wedge S) = M(R) \wedge M(S)$

(ii)  $M(R \vee S) = M(R) \vee M(S)$

*Proof:*

Since the relation meet and relation join matrices represents the intersection and union of the sets respectively, the result is so obvious.

*Illustration 4.2:*

Consider a set  $A = \{1,2,3\}$

Let  $R = \{(a,b) / a \mid b\}$  and

$S = \{(a,b) / a \leq b\}$  be two relations on  $A = \{1,2,3\}$ . Find  $M(R \wedge S)$  and  $M(R \vee S)$ .

*Solution:*

Given that R and S be two relation matrices on A whose matrices are

$$MR = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad MS = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Then } M(R \wedge S) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$M(R \vee S) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

*Illustration 4.3:*

Consider a set  $A = \{1,2,3\}$

Let  $T = \{(a,b) / a \geq b\}$  and

$U = \{(a,b) / a = b\}$  on  $A = \{1,2,3\}$ .

Find  $M(T \wedge U)$  and  $M(T \vee U)$ .

*Solution:*

The relations T and U can be represented by

$T = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$  and

$U = \{(1,1),(2,2),(3,3)\}$  whose matrices are

$$MT = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } MU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M(T \wedge U) = M(T) \wedge M(U) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M(T \vee U) = M(T) \vee M(U) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

*Illustration 4.4:*

Similarly, one can find the relation meet matrix and relation join matrix between the relations R and T, S and T, R and U, S and U.

*Theorem 4.2:*

Let R, S, T be three different relations on a set A. Then  $R \wedge (S \wedge T) = (R \wedge S) \wedge T$ .

*Proof:*

Let the Boolean matrices for the relations R, S, T is MR, MS, and MT respectively.

As shown in illustration 4.2 and 4.3, these Boolean matrices represent the respective relation meet matrices.

$$\begin{aligned} \text{We know that } M(R \wedge S) &= M(R) \wedge M(S) \\ \Rightarrow M(R \wedge (S \wedge T)) &= M(R) \wedge M(S \wedge T) \\ &= M(R) \wedge (M(S) \wedge M(T)) \end{aligned}$$

Similarly, we have

$$\begin{aligned} M((R \wedge S) \wedge T) &= M(R \wedge S) \wedge M(T) \\ &= (M(R) \wedge M(S)) \wedge M(T) \end{aligned}$$

Now, we know that the Boolean product is associative. This implies,

$$\begin{aligned} M(R) \wedge (M(S) \wedge M(T)) &= (M(R) \wedge M(S)) \wedge M(T) \\ M(R \wedge (S \wedge T)) &= M((R \wedge S) \wedge T) \end{aligned}$$

Since the Boolean matrices for these relations are the same, we have

$$R \wedge (S \wedge T) = (R \wedge S) \wedge T.$$

This completes the proof.

### **5. Determinant and Inverse of Relation Meet Matrix:**

Consider the relations R, S, T and U given in illustrations 4.2 and 4.3.

Let us try to find the determinant of relation meet matrices on these relations.

$$\begin{aligned} \text{Now } |M(R \wedge S)| &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= 1(0-0) - 1(0-0) + 1(0-0) = 1. \end{aligned}$$

$$\text{Similarly, } |M(R \wedge T)| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$|M(S \wedge T)| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$|M(R \wedge U)| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$|M(S \wedge U)| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1-0) - 0(0-0) + 0(0-0) = 1.$$



Hence, we can conclude that the determinant of relation meet matrix on posets is always equal to one.

∴ Inverse exists for every relation meet matrix on posets.

*Illustration 5.1 :*

Let  $R = \{(a,b) / a \mid b\}$  and  $S = \{(a,b) / a \leq b\}$  on  $A = \{1,2,3\}$ .

Find the inverse of relation meet matrix on R and S.

*Solution :*

The relations R and S are given by

$R = \{(1,1),(1,2),(1,3),(2,2),(3,3)\}$

$S = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$

$$MR = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad MS = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M(R \wedge S) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Also,  $|M(R \wedge S)| = 1$

$$(M(R \wedge S))^{-1} = \frac{1}{|M(R \wedge S)|} (\text{adj.}(M(R \wedge S)))$$

$$\text{adj.}(M(R \wedge S)) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore (M(R \wedge S))^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \because |M(R \wedge S)| = 1$$

*Illustration 5.2*

Let  $S = \{(a,b) / a \leq b\}$  and  $U = \{(a,b) / a = b\}$  on  $A = \{1,2,3\}$ .

Find the inverse of relation meet matrix on S and U.

*Solution :*

The relations S and U are given by

$S = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$

$U = \{(1,1),(2,2),(3,3)\}$

$$MS = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad MU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M(S \wedge U) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Also,  $|M(S \wedge U)| = 1$

$$(M(S \wedge U))^{-1} = \frac{1}{|M(S \wedge U)|} (\text{adj.}(M(S \wedge U)))$$

$$\text{adj}(M(S \wedge U)) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore (M(S \wedge U))^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \because |M(S \wedge U)| = 1$$

#### 6. Determinant and Inverse of Join Matrices:

Consider the relations S, T and U given in illustrations 4.2 and 4.3.

Let us try to find the determinant of relation join matrices on these relations.

$$\begin{aligned} \text{Now } |M(T \vee U)| &= \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \\ &= 1(1 \cdot 0) - 0(1 \cdot 0) + 0(1 \cdot 1) = 1. \end{aligned}$$

$$\text{Similarly, } |M(S \vee U)| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\therefore |M(T \vee U)| = |M(S \vee U)| = 1,$$

$\therefore$  Inverse exists for  $M(T \vee U)$  and  $M(S \vee U)$ .

But, this is not true in all cases of relation join matrices.

Consider the relation join matrix on S and T

$$|M(S \vee T)| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1(1 \cdot 1) - 1(1 \cdot 1) + (1 \cdot 1) = 0$$

In this situation, we can find generalised inverse<sup>9</sup> for  $M(S \vee T)$ .

[Let A be a matrix then G be the generalised inverse of A, if  $AGA = A$ ]

$$\text{Consider a } 3 \times 3 \text{ matrix } G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} (M(S \vee T)) G (M(S \vee T)) &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= M(S \vee T) \end{aligned}$$

$$\Rightarrow M(S \vee T) G M(S \vee T) = M(S \vee T)$$

$$\therefore G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is the generalised inverse of } M(S \vee T).$$

*Illustration 6.1 :*

Let  $T = \{(a,b) / a \geq b\}$  and  $U = \{(a,b) / a = b\}$  on  $A = \{1,2,3\}$ .  
Find the inverse of relation join matrix on  $T$  and  $U$ .

*Solution :*

The relations  $T$  and  $U$  are given by

$$T = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$$

$$U = \{(1,1),(2,2),(3,3)\}$$

$$MT = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } MU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M(T \vee U) = M(T) \vee M(U) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{Also, } |M(T \vee U)| = 1$$

$$(M(T \vee U))^{-1} = \frac{1}{|M(T \vee U)|} (\text{adj.}(M(T \vee U)))$$

$$\text{adj.}(M(T \vee U)) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\therefore (M(T \vee U))^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \because |M(T \vee U)| = 1$$

*Illustration 6.2 :*

Let  $S = \{(a, b) / a \leq b\}$  and  $U = \{(a, b) / a = b\}$  on  $A = \{1, 2, 3\}$ .  
Find the inverse of relation join matrix on  $S$  and  $U$ .

*Solution:*

The relations  $S$  and  $U$  are given by

$$S = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$$

$$U = \{(1, 1), (2, 2), (3, 3)\}$$

$$MS = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } MU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M(S \vee U) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Also, } |M(S \vee U)| = 1$$

$$(M(S \vee U))^{-1} = \frac{1}{|M(S \vee U)|} (\text{adj.}(M(S \vee U)))$$

$$\text{adj}(M(S \vee U)) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore (M(S \vee U))^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \because |M(SU)| = 1$$

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