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Regular number of subdivision of a graph

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Abstract

For any (p ,q) graph G,a subdivision graph S(G) is obtained from G by subdividing each edge of G. The regular number of a S(G) is the minimum number of subsets into which the edge set of S(G) should be partitioned so that the subgraph induced by each subset is regular and is denoted by $r_s(G)$. In this paper some results on regular number of $r_s(G)$ were obtained and expressed in terms of elements of G.

1. Introduction

All graphs considered here are simple, finite, non-trivial, with no loops and multiple edges. As usual p and q denote the number of vertices and edges of a graph G. As usual,the maximum degree of a vertex in G is denoted by $\Delta(G)$. A vertex v is called a cut vertex if removing it from G increases the number of components of G. A graph G is called trivial if it has no edges. The independence number $\beta_1(G)$ is the maximum cardinality of an edge independent set in G. The path and tree numbers were introduced by Stanton James and Cown in[2]. In this paper we denote p* and q* as the number of vertices and the number of edges in S(G). Any undefined term in this paper may be found in[1]. The edge set independence number $\beta_1^*(G)$ is the minimum order of partition of E(G) into subsets so that the subgraph induced by each set must be independent[1-2].

2. Results:

The following result is obvious, hence we omit its proof

Theorem 1 : For any graph G, $r_s(G)$ = 1, if and only if S(G) is regular.

Next we obtain the regular number of a wheel.

Theorem 2 : For any wheel W_p with $p \geq 5$ vertices,

$r_s(W_p) = p - 2$; if p is even,

$= \frac{p+3}{2}$; if p is odd.

Proof : Let $v_1, v_2, v_3, \ldots, v_p$, v_1', v_2' $v_3', \ldots, v_{p-1}'$, $v_1'', v_2'', v_3'' \ldots v_{p-1}''$, be the vertices of S($W_p$) such that deg v_i= 3 for 1 ≤ i ≤ p-1, deg v_j' = 2 for 1 ≤ j ≤ p-1 and deg v_k''= 2 for 1 ≤ k ≤ p-1 .

let $e_1, e_2, e_3, \ldots, e_{p-1}$, $e_1', e_2', e_3' \ldots$, $e_{p-1}', e_1'', e_2'', e_3'', \ldots, e_{p-1}''$, $e_1''', e_2''', e_3''', \ldots, e_{p-1}'''$ be the edges of S(W_p) such that $e_i = v_i v_i'$ for 1 ≤ i ≤ p-1 and $e_{p-1}' = v_1 v_{p-1}'$ $e_i' = v_i' v_{i+1}$ for 1 ≤ i ≤ p-1 $e_i'' = v_i v_i''$ for 1 ≤ i ≤ p-1 and $e_i''' = v_i'' v_p$ for 1 ≤ i ≤ p-1

then we consider the following two cases.

Case 1. If p is odd , then p-1 is even and hence,

$F_1 = \{e_1, e_1', e_2'', e_2''', e_1''', e_1''\}$, $F_2 = \{e_3, e_3', e_4'', e_4''', e_3''', e_3''\}$, $F_3 = \{e_5, e_5', e_6'', e_6''', e_5''', e_5''\}$, $F_4 = \{e_7, e_7', e_8'', e_8''', e_7''', e_7''\}$, $F_{n-1} = \{e_2', e_4', e_6', \ldots, e_{p-1}'\}$, $F_n = \{e_2, e_4, e_6, \ldots, e_{p-1}\}$ is the minimum regular partition of W_p.

Thus,

$$
\begin{aligned}
r_s(W_p) &= |\{F_1, F_2, F_3, \ldots, F_n\}| \\
&= |\{F_1, F_2, F_3, \ldots, F_{n-1}\}|+2 \\
&= \frac{p-1}{2} + 2 \\
&= \frac{p-1+4}{2} \\
&= \frac{p+3}{2}
\end{aligned}
$$

Case 2. If p is even then p-1 is odd and thus,

$F_1=\{e_1, e_1', e_1'', e_1''', e_2'', e_2'''\}$, $F_2=\{e_3, e_3', e_3'', e_3''', e_4'', e_4'''\}$, $F_3=\{e_5, e_5', e_5'', e_5''', e_6'', e_6'''\}$, $F_{n-2}=\{e_2, e_4, e_6, \ldots, e_{p-1}, e_{p-1}'''\}$, $F_{n-1}=\{e_2', e_4', e_6', \ldots, e_{p-1}'\}$, $F_n=\{e_{p-1}''\}$ is the minimum regular partitions of W_p. Hence,

$$
\begin{aligned}
r_s(W_p) &= |\{F_1, F_2, F_3, \ldots, F_n\}|- 2 \\
&= p - 2
\end{aligned}
$$

Hence the proof.

In the following theorem we establish the regular number of a regular graph.

Theorem 3: For every r-regular graph G with r ≥ 2, then $r_s(G) \leq$ r.

Proof : Let G be a r-regular graph with r ≥ 2. Let $V_1 = \{v_1, v_2, v_3, \ldots, v_p\}$ be the vertices of G with degree r and $V_2 = \{v_1', v_2', v_3', \ldots, v_q'\}$ be the vertices of S(G) such that $V_2 \subset V_s(G)$, $\forall\, v_i' \in V_2$ has degree two $V_s(G) = V_1 \cup V_2$. Let $E_s(G)=\{e_1, e_2, e_3, \ldots, e_r\}$ such that $e_1 = v_1 v_1'$, $e_2 = v_1 v_2'$, $\ldots$, $e_r = v_1 v_r'$ and $F_1 = \{e_1\}$ $F_2 = \{e_2\}, \ldots, F_r = \{e_r\}$

Hence $r_s(G) \leq |\{F_1, F_2, F_3, \ldots, F_r\}$, $r_s(G) \leq$ r.

Next we obtain the regular number for a tree.

Theorem 4. For any non-trivial tree T, $r_s(T)=\Delta$(T).

Proof : Let T be a non-trivial tree and F be a minimum regular partition of S(T). Now, we have S(T) in which v be a vertex of maximum degree in T. Since $v \in V_s(T)$ have same degree as in T. Let $u_1, u_2, u_3, \ldots u_n$, be the n-vertices adjacent to v. Since T is a tree, the subgraph induced by each subset of F is mK_2 with $m \geq 1$. Let $vu_1', vu_2', vu_3', \ldots, vu_n'$ and $u_1'u_1, u_2'u_2, u_3'u_3, \ldots, u_n'u_n$ be the edges of S(G). Here $vu_1', vu_2', vu_3', \ldots, vu_n'$ belongs to different sets of $F_1, F_2, F_3, \ldots F_n$ of respectively. Further each edge not incident to v belongs to any one of $F_1, F_2, F_3, \ldots F_n$.

Hence $r_s(T) = |F| = \Delta(T)$.

Lemma 5. For any path P_p, with $p \geq 2$, $r_s(P_p) = 2$.

Proof : In any path P_p, with $p \geq 2$, the maximum degree is 2

Hence, by the Theorem 4, we have, $r_s(P_p) = 2$.

Next we obtain the exact value of $r_s(G)$ for $G = K_{m,n}$ with $1 \leq m \leq n$.

Theorem 6. For any complete bipartite graph $K_{m,n}$ for $1 \leq m \leq n$,

$$r_s(K_{m,n}) = \frac{n}{m} \quad ; \text{if } n \equiv 0 \pmod{m}$$
$$= \left\lfloor \frac{n}{m} \right\rfloor + m \text{ ; if } n \equiv 1 \pmod{m}$$

Proof : Let $K_{m,n}$ be a complete bipartite graph with $1 \leq m \leq n$. Further we subdivide each edge of $K_{m,n}$ by a vertex of degree 2.

Then, we consider the following two cases.

Case 1. If $n \equiv 0 \pmod{m}$, then clearly $F = \{F_1, F_2, F_3, \ldots, F_j\}$ is the minimum regular partition of $S[K_{m,n}]$ such that the subgraph induced by each F_i $\langle F_i \rangle$ is $S[K_{m,n}]$ for $1 \leq i \leq j$

Thus,

$$r_s(K_{m,n}) = |F| = j = \frac{nm}{m^2} = \frac{n}{m}$$

Case 2. If $n \equiv 1 \pmod{m}$, then
$n-1 \equiv 0 \pmod{m}$
and hence,

$$r_s(K_{m,n}) = r_s(K_{m,n-1}) + r_s(K_{m-1}) = \frac{(n-1)}{m} + m = \left\lfloor \frac{n}{m} \right\rfloor + m.$$

In the next result we obtain a relationship between r(G) and $\beta_1^*[S(G)]$ of a graph G.

Theorem 7. For any graph G, $\beta_1^*[S(G)] \geq r(G)$.

Proof: Let $E(G) = \{e_1, e_2, e_3, \ldots, e_i\}$ be the edge set of G. Now we have the edge partition such that $r(G) = |F_1, F_2, F_3, \ldots, F_k|$. In S(G), S(G) = 2E(G) . Let $E'(G) = \{e_1', e_2', e_3', \ldots, e_i'\}$ be the edge subset in S(G) such that $E_s(G) = E(G) \cup E'(G)$.

Now, there exists a partition of $E_s(G)$ as $E_1(G) \cup E_1'(G)$, $E_1(G) \subset E(G)$ and $E_1(G) \subset E'(G)$ and the subgraph induced by each set is independent. Hence,

$|E_1(G)\ E_1'(G)| = \beta_1^*[S(G)]$ which gives $|E_1(G)\ E_1'(G)| \geq |F_1, F_2, F_3, \ldots, F_k|$

Clearly, $\beta_1^*[S(G)] \geq r(G)$.

In the following theorem we establish the bound in terms of edges and maximum independence number of G for maximum edge independence number of subdivision of G.

Theorem 8. For any graph G, $\beta_1^*[S(G)] \leq q - \beta_1(G)$.

*Proof :*Let S be a maximum edge independent set in G and E(G)={$e_1, e_2, e_3, \ldots, e_i$} be the edge set of G. And we have the edge partition such that S(G) = 2E(G). Then E –S has at most | E – S | edge independent sets.

Thus, $\beta_1^*[S(G)] \leq |E - S|$, $\beta_1^*[S(G)] \leq q - \beta_1(G)$.

Next we obtain the another result relationship between $r_s(G)$ and q^*.

Theorem 9. For any graph G, with q^* number of edges in S(G), then $r_s(G) \leq q^*$.

Proof: Let E(G) = {$e_1, e_2, e_3, \ldots \ldots \ldots e_i$} be the edge set of G. Suppose there exists a vertex v_j $1 \leq j \leq i$ with $j = 1$ and has maximum degree. Let $e_1, e_2, e_3, \ldots \ldots \ldots e_j$ be the number of edges incident to v_j. The subgraph induced by each subset of F is mK_2, $m \geq 1$.

In S(G) , $E_S(G) = E(G) \cup E'(G)$ and every edge of $E'(G)$ belongs to any one of F, such that they are disjoint. Further every edge of $E_S(G)$ belongs to different sets of $F_1, F_2, F_3, \ldots, F_j$ of F .

Hence, $|F| \leq |E(G) \cup E'(G)$, $r_s(G) \leq q^*$.

Next, we obtain the result on minimum set of edges.

Theorem 10: For any graph G, with subdivision S(G) of G, r[S(G)–X] = 1 if and only if X = V(G) – $V_s(G)$, where X is the minimum set of edges of G .

Proof : Let r[S(G) – X] = 1. Let G has S(G) with X number of vertices. Then clearly S(G)– X is either 1-regular or 0-regular. Since G has X number of edges which are isomorphic to V(G) – $V_s(G)$, then S(G)-X is not 1-regular.

Hence, S(G) –X is 0-regular. Thus, X=V(G)– $V_s(G)$.

Conversely, suppose X = V(G) – $V_s(G)$. Then X number of vertices divides each edge of G. clearly X is the number of edges in G.

Next, we obtain the result on regular graph and relationship between the cut vertex and blocks.

Theorem 11: For any graph G, with a cut vertex *v*, incident to {$b_1, b_2, b_3, \ldots, b_n$} $n \geq 2$ blocks which are complete and distinct, r(G-*v*) = n.

Proof: Let r(G-*v*) = n . Let *v* be a cut vertex incident to two or more blocks. Let H = {$b_1, b_2, b_3, \ldots, b_n$} be the blocks of G, then clearly r(G-*v*)=n or (n-1). Suppose, r(G-*v*) = n. We consider the following cases.

Case 1. Assume there exists a block $b_i \in$ H $1 \leq i \leq n$ and is not complete. Then for any positive integer k, r(G-v) = n-k, k ≥ 1 which is a contradiction.

Case 2. Assume there exists at least two blocks b_i, i = 1 , 2 which are isomorphic to each other. Then these $b_i \in F_1$. Hence r(G-v) < n-1. Which is a contradiction

Conversely, suppose all n-blocks are regular and distinct. Let $F_1 = \{b_1\}$, $F_2 = \{b_2\}$, $F_3 = \{b_3\}$, . . . , $F_n = \{b_n\}$ be the partition of G. Hence $|F_n|$ = n. Let v be a cut vertex of G incident with n, n≥ 2 blocks. Then in G - v, we have n-components in which each block is again complete. Clearly each component belongs to each set F_i, i = 1,2,3, ….., *n, such that* $\sum_{i=1}^{n} F_i =$ F , thus | F | = n and | F | = r(G-v) = n. Hence the proof.

In the next result we obtain the relationship between r_s(G) and the number of vertices of a graph.

Theorem 12: For any graph G, G≠ $K_{1,n}$, n ≥ 2 , then $r_s(G) \leq$ p – 2.

Proof: Suppose G has p ≤ 3 vertices. Assume G = P_3, then S(G) = P_5, then F_1 , F_2 are the minimum regular partition of S[P_3].

Hence,

$$r[S(P_3)] = |\{F_1, F_2\}| = 2 > p - 2.$$

Thus, we consider G ≠ P_3 and has p≥3 vertices.

Suppose G =$K_{1,p}$,p ≥ 2. Then V(G) = p +1 and E(G) = $e_1, e_2, e_3, \dots, e_n$. Since deg(($v$) = Δ(G) = p, $v \in V(G)$. In S(G), $E_s(G)$ $= e_1' e_1'' , e_2' e_2'', \dots , e_n' e_n''$. Where $e_i'\ e_i'' = e_i$ $1 \leq i \leq n$ and N(e_i') =e_i'' . since every e_i is incident to v in G, similarly every e_i' is also incident to v gives $F_1' , F_2' , F_3' , \dots , F_n'$
partitions. Similarly $e_1'' , e_2'' , e_3'' , \dots , e_n''$ gives another partition as F_1'' , $F_2'' , \dots , F_n''$ such that $|\{F_1' , F_2' , F_3' , \dots , F_n'\}| = |\{F_1'' , F_2'' , F_3'' , \dots , F_n''\}|$

Now, r_s(G) = p > p–2 hence G≠$K_{1,p}$, p≥2. Now to get r_s(G) , G may be any graph. We consider the following cases.

Case (i). Suppose G is a tree. Let G has $v_i, i \geq n$ and $v_i \in V(G)$ such that deg(v_i) = Δ(G)
Then,

Let $v_1 , v_2 , v_3 , \dots , v_n$ be the vertices with deg(v_i)≥2. Then Δ(G) has v_j number of vertices such that $1 \leq j \leq n$, clearly $F_1, F_2, F_3, \dots , F_j$ be the partition of edges incident to v_j and the remaining edges belongs to any one of the F_j. Hence $|\{F_1, F_2, F_3, \dots , F_j\}|$ = Δ(G).

In S[G], $E_s(G) = \{e_1' , e_2' , e_3' , \dots , e_n'\} \cup \{ e_1'' , e_2'' , e_3'' , \dots , e_n''\}$ and $\{e_j'\} \cup \{e_j''\} = \{F_1, F_2, F_3, \dots , F_j\}$

Thus, $|\{F_1, F_2, F_3, \dots , F_j\}|$ = p – 2.

Case (ii) : Suppose G is not a tree. Then there exists at least one block which is not an edge. Now for S[G], the same argument will exists as in case (i). Hence partition $\{F_1, F_2, F_3, \ldots, F_k\}$ be the partition of $E_s(G)$ such that $|\{F_1, F_2, F_3, \ldots, F_k\}| \leq p - 2$.

3. Conclusion

We established the regular number of subdivision of some standard graphs by subdividing. Further we developed the upper bound in terms of minimum edge independence number of G and vertices of G. Also many results established are sharp.

References

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