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On Quasi **g-Open and Quasi **g-Closed Functions

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Abstract

In this paper, we introduce a new type of open function namely quasi **g-open function. Further we obtain its characterizations and its basic properties.

Key words and phrases: **g-open set, **g-closed set, **g-interior, **g-closure, quasi **g-open function.

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1. Introduction and Preliminaries

Functions stand among the important notions in the whole of mathematical science. Many different open functions have been introduced over the years. The importance is significant in various area of mathematics and sciences. The notion of **g-closed sets were introduced and studied by Manoj *et al.*[5]. In this paper, we will continue the study of related functions by considering **g-open sets and **g-open functions. We further introduce and characterize the concept of quasi **g-open functions.

Throughout this paper, spaces mean topological spaces on which no separation axioms are assumed unless otherwise mentioned and $f : (X, \tau) \rightarrow (Y, \sigma)$ denotes a function f of a space (X, τ) into a space (Y, σ). Let A be a subset of space X. Then the closure and the interior of A are denoted by cl(A) and int(A) respectively.

Definition 1.1: A subset A of a topological space (X, τ) is called semi-open[2] (resp. semi- closed) if $A \subseteq cl(int(A))$ (resp. $int(cl(A)) \subseteq A$).

The semi-closure[1] of a subset A of X (denoted by scl(A)) is defined to be the intersection of all semi-closed sets containing A.

Definition 1.2: A subset A of a topological space (X, τ) is called

(i) sg-closed[3] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X. The complement of sg-closed set is called sg-open.

(ii) $\hat{\hat{g}}$-closed[4] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is sg-open in X. The complement of $\hat{\hat{g}}$-closed set is called $\hat{\hat{g}}$-open.

(iii) **g-closed[5] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\hat{\hat{g}}$-open in X. The complement of **g-closed set is called **g-open.

The union (resp. intersection) of all **g-open (resp. **g-closed) sets, each contained in (resp. containing) a set A in a space X is called the **g-interior (resp. **g-closure) of A and is denoted by **g-Int(A) (resp. **g-cl(A))[5].

Definition 1.3: A function f : (X, τ) $\rightarrow$ (Y, σ) is called

(i) **g-irresolute[5] (resp. **g-continuous[5]) if the inverse image of every **g-closed (resp. closed) set in Y is **g-closed in X.

(ii) **g-open[5] (resp. **g-closed[5]) if f(V) is **g-open (resp. **g-closed) in Y for every open (resp. closed) subset of X.

(iii) **g*-closed[5] if the image of every **g-closed subset of X is **g-closed in Y.

Definition 1.4: Let x be a point of (X, τ) and N be a subset of X. Then N is called[5] a **g-neighborhood (briefly **g-nbd) of x if there exists a **g-open set G such that x $\in$ G and G $\subset$ N.

*2. Quasi **g-open Functions :*

In this section we introduce the following definitions.

Definition 2.1: A function f : (X, τ) $\rightarrow$ (Y, σ) is called quasi **g-open if the image of every **g-open set in X is open in Y.

If the function f is bijective then the concept of quasi **g-openness and **g-continuity coincide.

Theorem 2.2: A function f : X $\rightarrow$ Y is quasi **g-open iff for every subset A of X, f(**g-Int(A)) $\subseteq$ Int(f(A)).

Proof: Suppose that f is quasi **g-open function. Since Int(A)) $\subseteq$ A and **g-Int (A) is a **g-open set so f(**g-Int(A)) $\subseteq$ (f(A)). As f(**g-Int(A)) is open, f(**g-Int(A)) $\subseteq$ Int(f(A)).

Conversely, Let A be **g-open set in X such that f(**g-Int(A)) $\subseteq$ Int(f(A)). Then f(A) = f(**g-Int(A)) $\subseteq$ Int(f(A)). But Int(f(A)) $\subseteq$ f(A) so f(A) = Int(f(A)) and hence f is quasi **g-open.

Lemma 2.3: If a function f : X $\rightarrow$ Y is quasi **g-open, then **g-Int(f^{-1}(A)) $\subseteq$ f^{-1}(Int(A)) for every subset A of Y.

Proof: Let A be any arbitrary subset of Y. Then **g-Int(f^{-1}(A)) is a **g-open set in X and f is quasi **g-open, then f(**g-Int(f^{-1}(A)) $\subseteq$ Int(f(f^{-1}(A)) $\subseteq$ Int(A). Thus **g-Int (f^{-1}(A)) $\subseteq$ f^{-1}(Int(A)).

Theorem 2.4: For a function f : X $\rightarrow$ Y, the following are equivalent

(i) f is quasi **g-open,

(ii) For each subset A of X, f(**g-Int(A))

$\subseteq$ Int(f(A)).

(iii) For each $x \in X$ and each **g-nbd A of x in X, there exists a neighborhood A of x in X, there exists a neighborhood B of f(x) in Y such that $B \subseteq f(A)$.

Proof: (i)$\Rightarrow$(ii) Follows from theorem (2.2).

(i) $\Rightarrow$ (iii) Let $x \in X$ and A be an arbitrary **g-nbd of x in X. Then there exists a **g-open set B in X such that $x \in B \subseteq A$. So by (ii), f(B) = f(**g-Int(B)) $\subseteq$ Int(f(B)) and hence f(B) = Int(f(B). Thus f(B) is open in Y such that $f(x) \in f(B) \subseteq f(A)$.

(ii) $\Rightarrow$ (i) Let A be an arbitrary **g-open set in X. Then for each $y \in f(A)$, by (iii) there exists a nbd. B_y of y in Y such that $B_y \subseteq f(A)$. Since B_y is a nbd. of y so there exists an open set C_y in Y such that $y \in C_y \subseteq B_y$. Thus $f(A) = \cup\{C_y : y \in f(A)$ which is an open set in Y. Thus f is quasi **g-open function.

Theorem 2.5 : A function $f : X \to Y$ is quasi **g-open iff for any subset B of Y and for any **g-closed set A of X containing $f^{-1}(B)$ there exists a closed set C of Y containing B such that $f^{-1}(C) \subseteq A$.

Proof: Suppose that f is quasi **g-open function. Let $B \subseteq Y$ and A be a **g-closed set of X containing $f^{-1}(B)$. Put $C = Y - f(X - A)$. Clearly $f^{-1}(B) \subseteq A$ implies $B \subseteq C$. Since f is quasi **g-open so C is a closed set of Y. Moreover $f^{-1}(C) \subseteq A$.

Conversely, let U be **g-open set in X. Put $B = Y - f(U)$ then $X - U$ is a **g-closed set in X containing $f^{-1}(B)$. By hypothesis, there exists a closed set A of Y such that $B \subseteq A$ and $f^{-1}(A) \subseteq X - U$. Hence $f(U) \subseteq Y - A$. Again $B \subseteq A$, $Y - A \subseteq Y - B = f(U)$. Thus $f(U) = Y - A$ which is open and hence f is a quasi **g-open function[4].

Theorem 2.6: A function $f : X \to Y$ is quasi **g-open iff for any subset $f^{-1}(cl(B)) \subseteq$ **g-cl($f^{-1}(B)$) for every subset B of Y.

Proof: Let f be a quasi **g-open function. For any subset B of Y, $f^{-1}(B) \subseteq$ **g-cl($f^{-1}(B)$). So by theorem (2.5), there exists a closed set A in Y such that $B \subseteq A$ and $f^{-1}(A) \subseteq$ **g-cl($f^{-1}(B)$). Thus $f^{-1}(cl(B)) \subseteq f^{-1}(A) \subseteq$ **g-cl($f^{-1}(B)$).

Conversely, let $B \subseteq Y$ and A be a **g-closed set of X containing $f^{-1}(B)$. Put $C = cl_Y(B)$, then $B \subseteq C$ and C is closed and $f^{-1}(C) \subseteq$ **g-cl($f^{-1}(B)$) $\subseteq A$. Thus by theorem (2.5), f is quasi **g-open.

Lemma 2.7: Let $f : X \to Y$ and $g : Y \to Z$ be two functions and $gof : X \to Z$ is quasi **g-open. If g is continuous injective, then f is quasi **g-open.

Proof: Let A be a **g-open set in X. Then (gof)(A) is open in Z since gof is quasi **g-open. Again g is an injective continuous function, $f(A) = g^{-1}(gof(A))$ is open in Y. Thus f is quasi **g-open.

*3. Quasi **g-closed Functions :*

In this section we introduce the following definitions.

Definition 3.1: A function $f : (X, \tau) \to (Y, \sigma)$ is called quasi **g-closed if the image of each **g-closed set in X is open in Y.

Clearly every quasi **g-closed function is closed and **g-closed.

Remark 3.2: Every **g-closed (resp. closed) function need not be quasi **g-closed as shown by the following example.

Example 3.3: Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a\}, \{c\}, \{a, c\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by identity mapping then f is **g-closed and closed but not quasi **g-closed.

Lemma 3.4: If a function $f : X \to Y$ is quasi **g-closed, then $f^{-1}(\text{Int}(A)) \subseteq$ **g-Int $(f^{-1}(A))$ for every subset A of Y.

Proof: See Lemma (2.3).

Theorem 3.5 : A function $f : X \to Y$ is quasi **g-closed iff for any subset A of Y and for any **g-open set G of X containing $f^{-1}(A)$, there exists an open set U of Y containing A such that $f^{-1}(U) \subseteq G$.

Proof: See theorem (2.5).

Theorem 3.6: If $f : X \to Y$ and $g : Y \to Z$ are two quasi **g-closed function, then their composition $gof : X \to Z$ is a quasi **g-closed function.

Proof: Proof is definition based.

Theorem 3.7: Let $f : X \to Y$ and $g : Y \to Z$ be any two functions then

(i) If f is **g-closed and g is quasi **g-closed, then gof is closed.
(ii) If f is quasi **g-closed and g is **g-closed, then gof is **g*-closed.
(iii) If f is **g*-closed and g is quasi **g-closed, then gof is quasi **g-closed.

Theorem 3.8: Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that their composition $gof : X \to Z$ is quasi **g-closed

(i) If f is **g-irresolute surjective, then g is closed.
(ii) If g is **g-continuous injective, then f is **g*-closed.

Proof: (i) Let F be an arbitrary closed set in Y. Since f is **g-irresolute, $f^{-1}(F)$ is **g-closed in X. Again since gof is quasi **g-closed and f is surjective, $(gof(f^{-1}(F))) = g(F)$ is closed set in Z. Thus g is closed function.

(ii) Let F be any **g-closed set in X. Since gof is quasi **g-closed, (gof)(F) is closed in Z. Again g is **g-continuous injective function, $g^{-1}(gof(F)) = f(F)$ is **g-closed in Y. Thus f is **g*-closed.

Theorem 3.9: Let X and Y be two topological spaces. Then the function $g : X \to Y$ is a quasi **g-closed if and only if g(X) is closed in Y and $g(V) \setminus g(X \setminus V)$ is open in g(X) whenever V is **g-open in X.

Proof: Let $g : X \to Y$ is a quasi **g-closed function. Since X is **g-closed g(X) is closed in Y and $g(V) \setminus g(X \setminus V) = g(V) \cap g(X) \setminus g(X \setminus V)$ is open in g(X) when V is **g-open in X.

Conversely, let g(X) is closed in Y, $g(V) \setminus$

$g(X \setminus V)$ is open in $g(X)$ when V is **g-open in X and let F be closed in X. Then $g(F) = g(X) \setminus (g(X \setminus F) \setminus g(F))$ is closed in $g(X)$ and hence, closed in Y.

Corollary 3.10: Let X and Y be two topological spaces. Then a surjection function $g : X \rightarrow Y$ is quasi **g-closed if and only if $g(V) \setminus g(X \setminus V)$ is open in Y whenever V is **g-open in X.

Corollary 3.11: Let X and Y be two topological spaces and let $g : X \rightarrow Y$ be a **g-continuous, quasi **g-closed surjective function. Then the topology on Y is $\{g(V) \setminus g(X \setminus V) : V$ is **g-open in $X\}$.

Proof: Let G be open in Y. Then $g^{-1}(G)$ is **g-open in X and $g(f^{-1}(G) \setminus g(X \setminus g^{-1}(G)) = G$. Hence all open sets in Y are of the form $g(V) \setminus g(X \setminus V)$, V is **g-open in X. Also all sets of the form $g(V) \setminus g(X \setminus V)$, V is **g-open in X, are open in Y from corollary (3.10).

Definition 3.12: A topological space (X, τ) is said to be **g-normal if for any pair of disjoint **g-closed subsets F_1 and F_2 of X, there exists disjoint open sets U and V such that $F_1 \subseteq U$ and $F_2 \subseteq V$.

Theorem 3.13: Let X and Y be topological spaces with X is **g-normal. If $g : X \rightarrow Y$ is a **g-continuous quasi **g-closed surjective function. Then Y is normal.

Proof: Let F_1 and F_2 be disjoint closed subsets of Y then $g^{-1}(F_1)$ and $g^{-1}(F_2)$ are disjoint **g-closed subsets of X. Since X is **g-normal, there exists disjoint open sets G_1 and G_2 such that $g^{-1}(F_1) \subseteq G_1$ and $g^{-1}(F_2) \subseteq G_2$. Then $F_1 \subseteq g(G_1) \setminus g(X \setminus G_1)$ and $F_2 \subseteq g(G_2) \setminus g(X \setminus G_2)$. Further by corollary (3.10), $g(G_1) \setminus g(X \setminus G_1)$ and $g(G_2) \setminus g(X \setminus G_2)$ are open sets in Y such that $g(G_1) \setminus g(X \setminus G_1) \cap g(G_2) \setminus g(X \setminus G_2) = \Phi$. Thus Y is normal.

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