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On the Weyl projective curvature tensor of the projective semi-symmetric connection in an SP-Sasakian manifold

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Abstract

The objective of the present paper is to study the W_2-curvature tensor of the projective semi-symmetric connection in an SP-Sasakian manifold. It is shown that an SP-Sasakian manifold satisfying the conditions $\tilde{P} \cdot \widetilde{W}_2 = 0$ is an Einstein manifold and $\widetilde{W}_2 \cdot \tilde{P} = 0$ is a quasi Einstein manifold.

Key words and phrases: Projective semi-symmetric connection, SP-Sasakian manifold, quasi-Einstein manifold, W_2-curvature tensor, Einstein manifold.

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1 Introduction

The study of semi-symmetric connections is a very attractive field for investigations in the past many decades. Semi-symmetric connection was introduced by A. Friedmann and J. A. Schouten[5] in 1924. In 1930, E. Bartolotti[3] extended a geometrical meaning to such a connection. Further, H. A. Hayden[6] studied a metric connection with torsion on Riemannian manifold. After a long gap, in 1970, the study of semi-symmetric connections was resumed by K. Yano[13]. In particular, he studied semi-symmetric metric connections. Afterwards several researchers have been carried out the study of semi-symmetric connections in a variety of directions such as [10, 11,16,17]. The studies on projective semi-symmetric connections have been further extended by P. Zhao[15], S.K. Pal[7] and others.

In this article, we consider the projective semi-symmetric connection on an SP-Sasakian manifold and study some properties of W_2-tensor fields. The paper is organised as follows: We present a brief account of SP-Sasakian manifold in section 2. The subsequent section 3 is devoted to the brief description of the projective semi-symmetric connection and its properties. In section 4, we study the W_2-curvature tensor of projective semi-symmetric connection in an SP-Sasakian manifold. In section 5, we consider the two condition $\tilde{P}\cdot\widetilde{W}_2 = 0$ and $\widetilde{W}_2\cdot\tilde{P} = 0$ respectively Einstein manifold and quasi Einstein manifold.

2 Preliminaries :

The notion of an (almost) para-contact manifold was introduced by I. Sato[9]. An n-dimensional differentiable manifold M is said to have almost para-contact structure (ϕ,ξ,η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field known as characteristic vector field and η is a -form satisfying the following relations

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.1}$$

$$\eta(\bar{X}) = 0, \tag{2.2}$$

$$\phi(\xi) = 0, \tag{2.3}$$

and

$$\eta(\xi) = 1. \tag{2.4}$$

A differentiable manifold with almost para-contact structure (ϕ,ξ,η) is called an almost para-contact manifold. Further, if the manifold M has a Riemannian metric g satisfying

$$\eta(X) = g(X,\xi), \tag{2.5}$$

and

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.6}$$

Then the set (ϕ,ξ,η,g) satisfying the conditions to is called an almost para-contact Riemannian structure and the manifold M with such a structure is called an almost para-contact Riemannian manifold [2,9].

Now, let (M,g) be an n-dimensional Riemannian manifold with a positive definite metric g admitting a $1-$form η which satisfies the conditions

$$(\nabla_X\eta)Y - (\nabla_Y\eta)X = 0 \tag{2.7}$$

and

$$(\nabla_X\nabla_Y\eta)(Z) = -g(X,Z)\eta(Y) - g(X,Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z), \tag{2.8}$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Moreover, If (M,g) admits a vector field ξ and a $(1,1)$ tensor field ϕ such that

$$g(X,\xi) = \eta(X), \quad \eta(\xi) = 1 \quad and \quad \nabla_X\xi = \phi(X), \tag{2.9}$$

then it can be easily verified that the manifold under consideration becomes an almost para-contact Riemannian manifold. Such a manifold is called a para-Sasakian manifold or briefly a P-Sasakian manifold[1]. It is a special case of almost para-contact Riemannian manifold introduced by I. Sato. It is known[1] that on a P-Sasakian manifold the following relations hold:

$$\eta(R(X,Y,Z)) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.10}$$

$$R(\xi,X,Y) = -R(X,\xi,Y) = \eta(Y)X - g(X,Y)\xi, \tag{2.11}$$

$$R(\xi,X,\xi) = X - \eta(X)\xi, \tag{2.12}$$

$$R(X,Y,\xi) = \eta(X)Y - \eta(Y)X, \tag{2.13}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.14}$$

$$Q\xi = -(n-1)\xi, \tag{2.15}$$

$$r = -n(n-1), \tag{2.16}$$

where r is the scalar curvature.

A P-Sasakian manifold satisfying

$$(\nabla_X\eta)(Y) = -g(X,Y) + \eta(X)\eta(Y) \tag{2.17}$$

is called an special para-Sasakian manifold or briefly an SP-Sasakian manifold[1]. In an SP-Sasakian manifold, we also have

$$'F(X,Y) = -g(X,Y) + \eta(X)\eta(Y). \tag{2.18}$$

where $'F(X,Y)$ refers to the fundamental 2-form of the manifold. On an SP-Sasakian manifold, we also have the following:

$$\phi X = -X + \eta(X)\xi. \tag{2.19}$$

Quasi Einstein manifolds, introduced by M.C. Chaki and R.K. Maity[4], are natural generalisations of Einstein manifolds. According to them, a non-flat Riemannian manifold (M,g) $(n > 2)$ is a quasi-Einstein manifold[4] if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the following condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{2.20}$$

for all vector fields X and Y, where a and b are scalars with $b \neq 0$. A is a non-zero 1-form such that

$$g(X,\xi) = A(X), \tag{2.21}$$

for all vector fields X and ξ being a unit vector.

The W_2 curvature tensor is defined by[8]

$$W_2(X,Y,Z) = R(X,Y,Z) + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX], \tag{2.22}$$

where Q is a Ricci tensor of type (1,1), i.e., $S(X,Y) = g(QX,Y)$; S being the type (0,2) Ricci tensor. The Weyl projective curvature tensor P is defined by[12]

$$P(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{2.23}$$

where S being the type (0,2) Ricci tensor.

3 Projective semi-symmetric connection :

In this section, we give a brief account of projective semi-symmetric connection and study it on a SP-Sasakian manifold.

A linear connection $\widetilde{\nabla}$ on an n-dimensional Riemannian manifold (M, g) is called a semi-symmetric connection[13], if its torsion tensor T given by

$$T(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y]$$

has the form

$$T(X,Y) = \pi(Y)X - \pi(X)Y. \tag{3.1}$$

where π is a 1-form associated with a vector field ρ, i.e.,

$$\pi(X) = g(X,\rho). \tag{3.2}$$

Further, a connection $\widetilde{\nabla}$ is a metric connection if it satisfies

$$(\widetilde{\nabla}_X g)(Y,Z) = 0. \tag{3.3}$$

If the geodesic with respect to $\widetilde{\nabla}$ are always consistent with those of the Levi-Civita connection ∇ on a Riemannian manifold, then $\widetilde{\nabla}$ is called a connection projectively equivalent to ∇. If $\widetilde{\nabla}$ is linear connection projective equivalent to ∇ as well as a semi-symmetric one, we call $\widetilde{\nabla}$ is called projective semi-symmetric connection[14].

Now, we consider a projective semi-symmetric connection $\widetilde{\nabla}$ introduced by P. Zhao and H. Song[14] given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \Psi(Y)X + \Psi(X)Y + \Phi(Y)X - \Phi(X)Y, \tag{3.4}$$

for arbitrary vector fields X and Y, where the 1-forms Ψ and Φ are given through the following relations:

$$\Psi(X) = \frac{n-1}{2(n+1)}\pi(X) \text{ and } \Phi(X) = \frac{1}{2}\pi(X). \tag{3.5}$$

It is easy to see that the equations (3.4) and (3.5) give us

$$(\widetilde{\nabla}_X g)(Y,Z) = \frac{1}{n+1}[2\pi(X)g(Y,Z) - n\pi(Y)g(X,Z) - n\pi(Z)g(X,Y)], \tag{3.6}$$

which shows that the connection $\widetilde{\nabla}$ given by (3.4) is a metric one.

We denote by $\widetilde{R}$ and R the curvature tensors of the manifold relative to the projective semi-symmetric connection connections $\widetilde{\nabla}$ and the Levi-Civita connection ∇. It is known that[14] that

$$\widetilde{R}(X,Y,Z) = R(X,Y,Z) + \alpha(X,Z)Y - \alpha(Y,Z)X + \beta(X,Y)Z, \tag{3.7}$$

where α and β are the tensors of type $(0,2)$ given by the following relations

$$\alpha(X,Y) = \Psi'(X,Y) + \Phi'(Y,X) - \Psi(X)\Phi(Y) - \Psi(Y)\Phi(X) \tag{3.8}$$

$$\beta(X,Y) = \Psi'(X,Y) - \Psi'(Y,X) + \Phi'(Y,X) - \Phi'(X,Y). \tag{3.9}$$

The tensors Ψ' and Φ' of type $(0,2)$ are defined by the following two relations.

$$\Psi'(X,Y) = (\nabla_X \Psi)(Y) - \Psi(X)\Psi(Y). \tag{3.10}$$

and

$$\Phi'(X,Y) = (\nabla_X \Phi)(Y) - \Phi(X)\Phi(Y). \tag{3.11}$$

Contraction of the vector field X in the equation (3.7) yields a relation between Ricci tensors of the manifold relative to the two connections $\tilde{\nabla}$ and ∇ which is given by

$$\tilde{S}(Y,Z) = S(Y,Z) + \beta(Y,Z) - (n-1)\alpha(Y,Z) \tag{3.12}$$

Also, from the above equation, we get the following equation relating scalar curvatures $\tilde{r}$ and r of the manifold with respect to the two connections $\tilde{\nabla}$ and ∇

$$\tilde{r} = r + b - (n-1)a. \tag{3.13}$$

where $b = \sum_{i=1}^{n} \beta(e_i, e_i)$ and $a = \sum_{i=1}^{n} \alpha(e_i, e_i)$.

In order to extend the studies of the projective semi-symmetric connection $\tilde{\nabla}$ on SP-Sasakian manifold, we identify the 1-form π of the connection $\tilde{\nabla}$ with the 1-form η of the P-Sasakian manifold. In view of this equality between π and η and the equations (3.5), we find that the expression (3.4) for the projective semi-symmetric connection $\tilde{\nabla}$ reduces to

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(c+1)\eta(Y)X + \frac{1}{2}(c-1)\eta(X)Y, \tag{3.14}$$

where the constant c is given by $c = \frac{n-1}{n+1}$. Now, it can be seen that

$$(\tilde{\nabla}_X \eta)(Y) = (\nabla_X \eta)(Y) - c\eta(X)\eta(Y). \tag{3.15}$$

On an SP-Sasakian manifold, we have $(\nabla_X \eta)(Y) = (\nabla_Y \eta)(X)$. Therefore, the above equation yields

$$(\tilde{\nabla}_X \eta)Y = (\tilde{\nabla}_Y \eta)X.$$

Thus, the connection $\tilde{\nabla}$ given by the equation (3.4) becomes *special projective semi-symmetric connection* studied by S.K. Pal *et. al.*[7]. It can also be verified very easily that for such a projective semi-symmetric connection the tensor β vanishes and the tensor α is symmetric, i.e.,

$$\beta(X,Y) = 0, \quad \text{and} \quad \alpha(X,Y) = \alpha(Y,X). \tag{3.16}$$

As a consequence of these, the expressions for curvature tensor, the tensor α , Ricci tensors and scalar curvatures given by (3.7), (3.8), (3.12) and (3.13) takes the following simpler forms

$$\tilde{R}(X,Y,Z) = R(X,Y,Z) + \alpha(X,Z)Y - \alpha(Y,Z)X, \tag{3.17}$$

$$\alpha(X,Y) = \mu(\nabla_X \eta)(Y) - \mu^2 \eta(X)\eta(Y), \tag{3.18}$$

$$\tilde{S}(Y,Z) = S(Y,Z) - (n-1)\alpha(Y,Z). \tag{3.19}$$

and

$$\tilde{r} = r - (n-1)a, \tag{3.20}$$

where $\mu = \frac{1}{2}(c+1)$. In view of the equations (3.16) and (3.19), it follows that the Ricci tensor $\tilde{S}(Y,Z)$ of the special projective semi-symmetric connection is symmetric.

Now, we derive some of the results concerning the tensor α which we shall need in subsequent sections. It is easy to see that in view of the equation (2.17), the tensor takes the following form

$$\alpha(X,Y) = -\mu g(X,Y) + \nu\eta(X)\eta(Y), \tag{3.21}$$

where we have put $\nu = \mu - \mu^2$. This, in view of the equations (2.4) and (2.5) and symmetry of tensor α, produces easily that

$$\alpha(\xi, Y) = \alpha(Y, \xi) = \lambda\eta(Y), \tag{3.22}$$

where by λ we mean $-\mu^2$.

Now, putting ξ for each of the vector fields X, Y and Z in the equation (3.17) and using the equations (2.11), (2.13) and (3.22), we obtain the followings:

$$\tilde{R}(\xi, Y, Z) = \lambda^{'}\eta(Z)Y - \theta(Y, Z)\xi, \tag{3.23}$$

$$\tilde{R}(X, \xi, Z) = \theta(X, Z)\xi - \lambda^{'}\eta(Z)X, \tag{3.24}$$

and

$$\tilde{R}(X, Y, \xi) = \lambda^{'}\eta(X)Y - \lambda^{'}\eta(Y)X, \tag{3.25}$$

where $\lambda^{'} = (1 + \lambda)$ and the θ is a type (0,2) symmetric tensor given by

$$\theta(Y, Z) = g(Y, Z) + \alpha(Y, Z). \tag{3.26}$$

Also, using the equation (3.22) in the above, we at once get

$$\theta(Y, \xi) = \lambda^{'}\eta(Y). \tag{3.27}$$

On account of the equation (3.21), the expressions (3.17)and (3.19) for curvature tensor and Ricci tensor assumes the following forms:

$$\begin{aligned}\tilde{R}(X, Y, Z) &= R(X, Y, Z) - \mu[g(X, Z)Y - g(Y, Z)X] \\ &+\nu[\eta(X)Y - \eta(Y)X]\eta(Z),\end{aligned} \tag{3.28}$$

and

$$\tilde{S}(Y, Z) = S(Y, Z) + (n - 1)[\mu g(Y, Z) - \nu\eta(Y)\eta(Z)], \tag{3.29}$$

which produces the following expression for (1,1) type Ricci tensor Q

$$\tilde{Q}Y = QY + \mu(n - 1)Y - \nu(n - 1)\eta(Y)\xi. \tag{3.30}$$

Also, contraction of the equation (3.29) yields

$$\tilde{r} = r + (n - 1)(n\mu - \nu). \tag{3.31}$$

Taking inner product of the equation (3.17)with , we get the following

$$\eta(\tilde{R}(X, Y, Z)) = \eta(R(X, Y, Z)) + \alpha(X, Z)\eta(Y) - \alpha(Y, Z)\eta(X), \tag{3.32}$$

which, due to the equations (2.10) and (3.21), gives

$$\eta(\tilde{R}(X, Y, Z)) = -(\mu - 1)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]. \tag{3.33}$$

In (3.29) equation put $Z = \xi$ and using (2.14), (2.9) we get

$$\tilde{S}(Y, \xi) = (\mu - \nu - 1)(n - 1)\eta(Y). \tag{3.34}$$

4 *The W_2 Curvature tensor :*

In this section, we consider the W_2 curvature tensor of projective semi-symmetric connection in an SP-Sasakian manifold.

The W_2 curvature tensor field in an SP-Sasakian manifold is given by the following relation:

$$W_2(X, Y, Z) = R(X, Y, Z) + \frac{1}{n - 1}[g(X, Z)QY - g(Y, Z)QX]. \tag{4.1}$$

If we put ξ for X, Y and Z respectively in the above equation, then in view of the equations (2.9), (2.11), (2.13) and (2.15), we get

$$W_2(\xi, Y, Z) = \eta(Z)Y + \frac{1}{n-1}\eta(Z)QY, \tag{4.2}$$

$$W_2(X, \xi, Z) = -\eta(Z)X - \frac{1}{n-1}\eta(Z)QX \tag{4.3}$$

and

$$W_2(X, Y, \xi) = \eta(X)Y - \eta(Y)X + \frac{1}{n-1}[\eta(X)QY - \eta(Y)QX]. \tag{4.4}$$

Similar to the definition (4.1), we define the $\widetilde{W}_2$ curvature tensor of projective semi-symmetric connection $\widetilde{\nabla}$ in SP-Sasakian manifold by

$$\widetilde{W}_2(X, Y, Z) = \tilde{R}(X, Y, Z) + \frac{1}{n-1}[g(X, Z)\tilde{Q}Y - g(Y, Z)\tilde{Q}X], \tag{4.5}$$

Also, the type (0,4) tensor field $'\widetilde{W}_2$ is given by

$$'\widetilde{W}_2(X, Y, Z, U) = {}'\tilde{R}(X, Y, Z, U) + \frac{1}{n-1}[g(X, Z)\tilde{S}(Y, U) - g(Y, Z)\tilde{S}(X, U)], \tag{4.6}$$

where we have put

$$'\widetilde{W}_2(X, Y, Z, U) = g(\widetilde{W}_2(X, Y, Z), U)$$

and

$$'\tilde{R}(X, Y, Z, U) = g(\tilde{R}(X, Y, Z), U)$$

for the arbitrary vector fields X, Y, Z, U.

With the help of (3.17), (3.30) in (4.5), we get

$$\begin{aligned} \widetilde{W}_2(Y, Z, W) &= R(Y, Z, W) + \alpha(Y, W)Z - \alpha(Z, W)Y + \frac{1}{(n-1)} \\ &[g(Y, W)\{QZ - \nu(n-1)\eta(Z)\xi + \mu(n-1)Z\} \\ &-g(Z, W)\{QY - \nu(n-1)\eta(Y)\xi + \mu(n-1)Y\}], \end{aligned} \tag{4.7}$$

which using the equations (3.21) and (4.1), yields

$$\begin{aligned} \widetilde{W}_2(Y, Z, W) &= W_2(Y, Z, W) + \nu\{\eta(Y)Z - \eta(Z)Y\}\eta(W) \\ &+\nu\{\eta(Y)g(Z, W) - \eta(Z)g(Y, W)\}\xi. \end{aligned} \tag{4.8}$$

Now, taking ξ for each of the vector field Y, Z and W in the above equation and using (4.2), (4.3) and (4.4), we get

$$\begin{aligned} \widetilde{W}_2(\xi, Z, W) &= (1+\nu)\eta(W)Z + \frac{1}{n-1}\eta(W)QZ \\ &-2\nu\eta(Z)\eta(W)\xi + \nu g(Z, W)\xi, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \widetilde{W}_2(Y, \xi, W) &= -(1+\nu)\eta(W)Y - \frac{1}{n-1}\eta(W)QY \\ &+2\nu\eta(Y)\eta(W)\xi - \nu g(Y, W)\xi \end{aligned} \tag{4.10}$$

and

$$\widetilde{W}_2(Y,Z,\xi) = (1+\nu)\eta(Y)Z - (1+\nu)\eta(Z)Y + \frac{1}{n-1}\{\eta(Y)QZ - \eta(Z)QY\}. \tag{4.11}$$

Taking inner product with ξ in the equation (4.1) and using (2.14) and (2.10), we get

$$\eta(W_2(Y,Z,W)) = 0. \tag{4.12}$$

Similarly, from the equations (2.14), (3.33), (4.5) and (4.8), we obtain

$$\eta(\widetilde{W}_2(Y,Z,W)) = \nu\eta(Y)g(Z,W) - \nu\eta(Z)g(Y,W). \tag{4.13}$$

5 The Weyl Projective Curvature tensor :

In this section, we consider the Weyl projective curvature tensor of projective semi-symmetric connetion in an SP-Sasakian manifold and derive two theorems related to curvature conditions of semi-symmetry type. The Weyl projective curvature tensor in an SP-Sasakian manifold is given by the following relation:

$$P(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{5.1}$$

Similar to the definition (5.1), we define the Weyl projective curvature tensor of projective semi-symmetric connection $\widetilde{\nabla}$ in SP-Sasakian manifold by

$$\tilde{P}(X,Y,Z) = \tilde{R}(X,Y,Z) - \frac{1}{n-1}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y]. \tag{5.2}$$

Also the type (0,4) tensor field $'\tilde{P}$ is defined by

$$'\tilde{P}(X,Y,Z,U) = {}'\tilde{R}(X,Y,Z,U) - \frac{1}{n-1}[g(X,U)\tilde{S}(Y,Z) - g(Y,U)\tilde{S}(X,Z)], \tag{5.3}$$

where we have put

$$'\tilde{P}(X,Y,Z,U) = g(\tilde{P}(X,Y,Z),U)$$

and

$$'\tilde{R}(X,Y,Z,U) = g(\tilde{R}(X,Y,Z),U)$$

for the arbitrary vector fields X, Y, Z, U.

With the help of (3.21), (3.17), (3.29) and (5.1), we get

$$\tilde{P}(Y,Z,W) = P(Y,Z,W) = R(Y,Z,W) - \frac{1}{n-1}[S(Z,W)Y - S(Y,W)Z]. \tag{5.4}$$

Now taking ξ for each of the vector fields Y, Z and W in the above equation and using (2.11), (2.14), (2.13), we get

$$\tilde{P}(\xi,Z,W) = -g(Z,W)\xi - \frac{1}{(n-1)}S(Z,W)\xi, \tag{5.5}$$

$$\tilde{P}(Y,\xi,W) = g(Y,W)\xi + \frac{1}{(n-1)}S(Y,W)\xi \tag{5.6}$$

and

$$\tilde{P}(Y,Z,\xi) = 0. \tag{5.7}$$

Taking inner product with ξ in the equation (5.1) and using (2.5), (2.10), we get

$$\eta(P(Y,Z,W)) = \eta(Z)[g(Y,W) + \frac{1}{n-1}S(Y,W)] - \eta(Y)[g(Z,W) + \frac{1}{n-1}S(Z,W)]. \tag{5.8}$$

Similarly, from the equations (2.5), (3.33), (3.29) and (5.2), we obtain

$$\eta(\tilde{P}(Y,Z,W)) = g(Y,W)\eta(Z) - g(Z,W)\eta(Y) + \frac{1}{n-1}[S(Y,W)\eta(Z) - S(Z,W)\eta(Y)]. \tag{5.9}$$

Theorem 5.1 : If an SP*-Sasakian manifold admitting a projective semi-symmetric connection* $\tilde{\nabla}$ *satisfies* $\widetilde{W}_2(\xi,X)\cdot\tilde{P} = 0$*, then it is a quasi Einstein manifold.*

Proof : Let

$$(\widetilde{W}_2(\xi,X)\cdot\tilde{P})(Y,Z,W) = 0.$$

Therefore, we get from the above equation

$$\begin{aligned} 0 &= \widetilde{W}_2(\xi,X,\tilde{P}(Y,Z,W)) - \tilde{P}(\widetilde{W}_2(\xi,X,Y),Z,W) \\ &\quad -\tilde{P}(Y,\widetilde{W}_2(\xi,X,Z),W) - \tilde{P}(Y,Z,\widetilde{W}_2(\xi,X,W)). \end{aligned}$$

Now using (4.9), (5.5), (5.6), (5.7), (5.9) in the above equation, then taking inner product with ξ in the equation and using the equations (2.14), (5.4), (5.9), we get

$$\begin{aligned} '\tilde{R}(Y,Z,W,X)\nu &= -\frac{1}{(n-1)}\eta(W)\eta(Z)S(X,Y) + \frac{1}{(n-1)}\eta(W)\eta(Y)S(X,Z) \\ &+ \frac{1}{(n-1)}(2+\nu)S(X,Y)\eta(W)\eta(Z) - \frac{1}{(n-1)}(2+\nu)S(X,Z)\eta(W)\eta(Y) \\ &-(1+\nu)\eta(W)\eta(Y)g(Z,X) + \nu g(X,Z)g(Y,W) - \nu g(Z,W)g(X,Y) \\ &+(1+\nu)\eta(W)\eta(Z)g(X,Y). \end{aligned}$$

Now, contracting the above equation with respect to X and Y, we obtain

$$S(Z,W) = (1-n)g(Z,W) + \frac{(1-n)}{\nu}\eta(W)\eta(Z).$$

which proves that the manifold is a quasi Einstein manifold.

Theorem 5.2 : If an SP*-Sasakian manifold admitting a projective semi-symmetric connection* $\tilde{\nabla}$ *satisfies* $\tilde{P}(\xi,X)\cdot\widetilde{W}_2 = 0$*, then it is an Einstein manifold.*

Proof : Let

$$(\tilde{P}(\xi,X)\cdot\widetilde{W}_2)(Y,Z,W) = 0.$$

Then we have

$$0 = \tilde{P}(\xi, X, \widetilde{W}_2(Y, Z, W)) - \widetilde{W}_2(\tilde{P}(\xi, X, Y), Z, W) - \widetilde{W}_2(Y, \tilde{P}(\xi, X, Z), W) - \widetilde{W}_2(Y, Z, \tilde{P}(\xi, X, W)).$$

Now using (4.9), (4.10), (4.11) and (5.5) in the above equation, then taking inner product with ξ in the above equation and using the equations (3.17), (3.29), (4.6) and (4.8), we get

$$g(X, R(Y, Z, W)) + \alpha(Y, W)g(X, Z) - \alpha(Z, W)g(X, Y) + \frac{1}{(n-1)}g(Y, W)S(Z, X)$$
$$+\nu\eta(Z)\eta(X)g(Y, W) + g(W, Y)[\mu g(X, Z) - \nu\eta(Z)\eta(X)] - \frac{1}{(n-1)}g(Z, W)S(Y, X)$$
$$-\nu\eta(Y)\eta(X)g(Z, W) - g(W, Z)[\mu g(Y, X) - \nu\eta(Y)\eta(X)] + \frac{1}{(n-1)}[S(X, W_2(Y, Z, W)]$$
$$= -\nu\eta(Z)\eta(W)g(X, Y) + \nu g(X, Y)g(Z, W) + \frac{1}{(n-1)}\nu S(X, Y)g(Z, W)$$
$$+\nu\eta(Y)\eta(W)g(X, Z) - \nu g(X, Z)g(Y, W) - \frac{1}{(n-1)}\nu g(Y, W)S(Z, X).$$

Putting $Y = \xi$ in the above equation and using (2.11), (3.22), (4.2), we obtain

$$S(X, Z) = \frac{-(1 + \mu + \lambda)}{(1 + \nu)}g(X, Z)$$

which proves that the manifold is an Einstein manifold.

6 Scope of Future Research and Applications :

The study on the W_2- curvature tensor of the projective semi-symmetric connection in an *SP*-Sasakian manifold is our future research in this topic, which we are going to be published very soon.

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