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On Schur Complements In Bicomplex Representation Of q EP

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Abstract

It is established that under certain conditions a Schur Complement in bicomplex representation of q-EP is as well a q-EP matrix. As an application a decomposition of a partitioned matrix into a sum of bicomplex representation of q-EP matrices is given.

Key words : q-EP matrix, Schur Complements in q-EP, Schur Complements in Bicomplex representation of q-EP.

AMS Classification : 15A57, 15A15, 15A09

Introduction

Through we shall deal with $n \times n$ quaternion matrices: Let A^* denote the conjugate transpose of A. Any matrix $A \in H_{n\times n}$ is called q-EP. If $R(A) = R(A^*)$ and is called q-EP$_r$, if is q-EP [4] $(Q_E{}^{n\times n})$ and $rk(A) = r$, where $N(A)$, $R(A)$ and $rk(A)$ denote the null space, range space and rank of A respectively. It is well known that sum and product of q-EP, Generalized Inverse Group Inverse and Reverse order law for q-EP and Bicomplex representation methods and application of q-EP matrices.[5-9]

For any q-EP matrix , A can be uniquely represented as

$$A = A_0 + A_1 j \qquad \text{[by 8]}$$

$$R(A) = R(A_0) + R(A_1 j) \qquad \text{[by 5, Theorem 1]}$$

Where $A_s \in C_{n\times n}$ (s=0,1), $A_1 j$ means to multiply each entries of A_1 by j from right hand side and rk(A_0) = rk(A_1j).

In this section, Schur complements in bicomplex representation of q-EP matrices.

Lemma 1.1 :

If X and Y are generalized inverse of $A = A_0 + A_1 j$, then

$(C_0 + C_1 j) + (B_0 + B_1 j) = (C_0 + C_1 j)Y(B_0 + B_1 j)$ if and only if $N(A_0 + A_1 j) \subseteq M(C_0 + C_1 j)$ and $N(A_0 + A_1 j^*) \subseteq N(B_0 + B_1 j^*)$ or, equivalently if and only if

$$C = (C_0 + C_1 j)(A_0 + A_1 j)^-(A_0 + A_1 j) \text{ and } B = (A_0 + A_1 j)(A_0 + A_1 j)^- B \text{ for every } (A_0 + A_1 j)^- \quad (1)$$

. Throughout this paper, we are concerned with $n \times n$ quaternion matrices M partitioned in the form $M = M_0 + M_1 j$ where,

$$M_0 + M_1 j = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{pmatrix} A_0 + A_1 j & B_0 + B_1 j \\ C_0 + C_1 j & D_0 + D_1 j \end{pmatrix} \quad (2)$$

Where $A_0 + A_1 j$ and $D_0 + D_1 j$ are square matrices with respect to this partitioning a Schur complements of A in M is a matrix at the form $((M_0 + M_1 j)/(A_0 + A_1 j)) = (D_0 + D_1 j) - (C_0 + C_1 j)(A_0 + A_1 j)^-(B_0 + B_1 j)$. For entries of Schur complements one may refer to[2,3,5].

On account of lemma 1 it is obvious that under certain conditions $(M_0 + M_1 j)/(A_0 + A_1 j)$ is independent of the choice of $(A_0 + A_1 j)^-$. However in the sequel we shall always assume that $(M_0 + M_1 j)/(A_0 + A_1 j)$ is given in terms of specific choice of $(A_0 + A_1 j)^-$.

In[9] necessary and sufficient conditions are derived for a matrix of the (2) with and $C_0 + C_1 j = 0$ to be q-EP. The results are here extended for general matrices of the form (2). If a partitioned matrix of the form (2) is q-EP, then in general $((M_0 + M_1 j)/(A_0 + A_1 j))$ is not q-EP. Here we determine necessary and sufficient conditions for $(M_0 + M_1 j)/(A_0 + A_1 j)$ to be q-EP. In particular, when $rk(M_0 + M_1 j) = rk(A_0 + A_1 j)$ our results include as special cases the results of paper[14]. In[5] we have given conditions for a sum of q-EP matrices to be q-EP.

Theorem 1.2:

Let M be a matrix of the form (2) with

$N(A_0 + A_1 j) \subseteq N(C_0 + C_1 j)$ and $N(M_0 + M_1 j)/(A_0 + A_1 j) \subseteq N(B_0 + B_1 j)$, then the following are equivalent.

i. $M_0 + M_1 j$ is a q-EP matrix

ii. $A_0 + A_1 j$ and $(M_0 + M_1 j)/(A_0 + A_1 j)$ are q-EP, $N(A_0 + A_1 j)^* \subseteq N(B_0 + B_1 j)^*$ and $N((M_0 + M_1 j)/(A_0 + A_1 j)^*) \subseteq M(C_0 + C_1 j)^*$;

iii. Both the matrices

$$\begin{pmatrix} A_0 + A_1 j & 0 \\ C_0 + C_1 j & (M_0 + M_1 j)/(A_0 + A_1 j) \end{pmatrix} \text{ and } \begin{pmatrix} A_0 + A_1 j & B_0 + B_1 j \\ 0 & (M_0 + M_1 j)/(A_0 + A_1 j) \end{pmatrix}$$

are q-EP.

Proof :

$(i) \Rightarrow (ii)$

Let us consider the matrices

$$P = \begin{pmatrix} I_0 + I_1 j & 0 \\ (C_0 + C_1 j)(A_0 + A_1 j)^- & I_0 + I_1 j \end{pmatrix}, \; Q = \begin{pmatrix} I_0 + I_1 j & B_0 + B_1 j((M_0 + M_1 j)/(A_0 + A_1 j))^- \\ 0 & I_0 + I_1 j \end{pmatrix},$$

$$L = \begin{pmatrix} A_0 + A_1 j & 0 \\ 0 & (M_0 + M_1 j)/(A_0 + A_1 j) \end{pmatrix}$$

Clearly P and Q are non-singular. By assumption $N(A_0 + A_1 j) \subseteq N(C_0 + C_1 j)$ and $N(M_0 + M_1 j)/(A_0 + A_1 j) \subseteq N(B_0 + B_1 j)$ and by using Lemma 1.1 it is obvious that $M_0 + M_1 j$ can be factorized as $(M_0 + M_1 j) = PQL$. Hence $rk(M_0 + M_1 j) = rk(L_0 + L_1 j)$ and $N(M_0 + M_1 j) = N(L_0 + L_1 j)$.

But $M_0 + M_1 j$ is q-EP, e.g. $N(M_0 + M_1 j)^* = N(M_0 + M_1 j) = N(L_0 + L_1 j)$. Therefore by using Lemma 1.1 again $(M_0 + M_1 j)^* = (M_0 + M_1 j)^* (L_0 + L_1 j)^- (L_0 + L_1 j)$ holds for every $(L_0 + L_1 j)^-$. One choice of $(L_0 + L_1 j)^-$ is

$$(L_0 + L_1 j)^- = \begin{pmatrix} A_0 + A_1 j^- & 0 \\ 0 & ((M_0 + M_1 j)/(A_0 + A_1 j))^- \end{pmatrix}$$, which gives

$$(M_0 + M_1 j)^* = \begin{pmatrix} (A_0 + A_1 j)^* & (C_0 + C_1 j)^* \\ (B_0 + B_1 j)^* & (D_0 + D_1 j)^* \end{pmatrix}$$

$$= \begin{pmatrix} (A_0 + A_1 j)^* & (C_0 + C_1 j)^* \\ (B_0 + B_1 j)^* & (D_0 + D_1 j)^* \end{pmatrix} \begin{pmatrix} (A_0 + A_1 j)^- (A_0 + A_1 j) & 0 \\ 0 & ((M_0 + M_1 j)/(A_0 + A_1 j))^- ((M_0 + M_1 j)/(A_0 + A_1 j)) \end{pmatrix}$$

$(A_0 + A_1 j)^* = (A_0 + A_1 j)^* (A_0 + A_1 j)^- (A_0 + A_1 j)$ implies $N(A_0 + A_1 j)^* \supseteq N(A_0 + A_1 j)$, and since $rk(A_0 + A_1 j)^* = rk(A_0 + A_1 j)$ these imply $N(A_0 + A_1 j)^* = N(A_0 + A_1 j)$. Hence $A_0 + A_1 j$ is q-EP. From $(B_0 + B_1 j)^* = (B_0 + B_1 j)^* (A_0 + A_1 j)^- (A_0 + A_1 j)$ it follows that $N(B_0 + B_1 j) \supseteq N(A_0 + A_1 j) = N(A_0 + A_1 j)$.

After substituting $D_0 + D_1 j = (M_0 + M_1 j)/(A_0 + A_1 j) + ((B_0 + B_1 j)(A_0 + A_1 j)^- (C_0 + C_1 j))$ and using

$(C_0 + C_1 j)^* = (C_0 + C_1 j)^* ((M_0 + M_1 j)/(A_0 + A_1 j))^- ((M_0 + M_1 j)/(A_0 + A_1 j))$ in

$(D_0 + D_1 j)^* = (D_0 + D_1 j)^* ((M_0 + M_1 j)/(A_0 + A_1 j))^- (M_0 + M_1 j)/(A_0 + A_1 j)$ we get

$$(M_0 + M_1 j/A_0 + A_1 j)^* = (M_0 + M_1 j/A_0 + A1 j)^* (M_0 + M_1 j/A_0 + A_1 j)^- (M_0 + M_1 j)/(A_0 + A_1 j)$$

This implies that $N(((M_0 + M_1 j)/(A_0 + A_1 j))^*) \supseteq N((M_0 + M_1 j)/(A_0 + A_1 j))$ and since

$$rk((M_0 + M_1 j)/(A_0 + A_1 j))^* = rk((M_0 + M_1 j)/(A_0 + A_1 j))$$

we get $N((M_0 + M_1 j)/(A_0 + A_1 j))^* = N((M_0 + M_1 j)/(A_0 + A_1 j))$

Thus $(M_0 + M_1 j)/(A_0 + A_1 j)$ is q-EP. Further

$$N(C_0 + C_1 j)^* \supseteq N((M_0 + M_1 j)/(A_0 + A_1 j)) = N((M_0 + M_1 j)/(A_0 + A_1 j))^*$$

Hence (ii) holds.

$(i) \Rightarrow (ii)$. Since $N(A_0 + A_1 j) \subseteq N(C_0 + C_1 j)$, $N(A_0 + A_1 j)^* \subseteq N(B_0 + B_1 j)^*$, $N((M_0 + M_1 j)/(A_0 + A_1 j)) \subseteq N(B_0 + B_1 j)$ and $N((M_0 + M_1 j)/(A_0 + A_1 j))^* \subseteq N(C_0 + C_1 j)^*$ hold according to the assumption. So $(M_0 + M_1 j)^\dagger$ is given buy the formula

$$(M_0 + M_1 j)^\dagger = \begin{pmatrix} (A_0 + A_1 j)^\dagger + (A_0 + A_1 j)^\dagger (B_0 + B_1 j)((M_0 + M_1 j)/(A_0 + A_1 j))^\dagger (C_0 + C_1 j)(A_0 + A_1 j)^\dagger & -(A_0 + A_1 j)^\dagger (B_0 + B_1 j)((M_0 + M_1 j)/(A_0 + A_1 j))^\dagger \\ -((M_0 + M_1 j)/(A_0 + A_1 j))^\dagger ((C_0 + C_1 j)(A_0 + A_1 j))^\dagger & ((M_0 + M_1 j)/(A_0 + A_1 j))^\dagger \end{pmatrix}$$

According to lemma 1.1 the assumptions $N(A_0 + A_1 j) \subseteq N(C_0 + C_1 j)$ and $N(A_0 + A_1 j)^* \subseteq N(B_0 + B_1 j)^*$ imply that $(M_0 + M_1 j)/(A_0 + A_1 j)$ is invariant for every choice of $(A_0 + A_1 j)^-$. Hence $(M_0 + M_1 j)/(A_0 + A_1 j) = (D_0 + D_1 j) - ((C_0 + C_1 j)(A_0 + A_1 j))^\dagger (B_0 + B_1 j)$. Further, using $(C_0 + C_1 j) = ((M_0 + M_1 j)/(A_0 + A_1 j))((M_0 + M_1 j)/(A_0 + A_1 j))^\dagger (C_0 + C_1 j)$ and $(B_0 + B_1 j) = (A_0 + A_1 j)(A_0 + A_1 j)^\dagger (B_0 + B_1 j)$, $(M_0 + M_1 j)(M_0 + M_1 j)^\dagger$ is reduced to the form

$$(M_0+M_1j)^{\dagger}(M_0+M_1j)=\begin{pmatrix}(A_0+A_1j)(A_0+A_1j)^{\dagger} & 0\\ 0 & ((M_0+M_1j)/(A_0+A_1j))((M_0+M_1j)/((A_0+A_1j))^{\dagger}\end{pmatrix}$$

The relations $(A_0+A_1j)(A_0+A_1j)^{\dagger}=(A_0+A_1j)^{\dagger}(A_0+A_1j)$ and

$((M_0+M_1j)/(A_0+A_1j))((M_0+M_1j)/(A_0+A_1j))^{\dagger}=((M_0+M_1j)/(A_0+A_1j))^{\dagger}((M_0+M_1j)/(A_0+A_1j))$

result $(M_0+M_1j)(M_0+M_1j)^{\dagger}=(M_0+M_1j)^{\dagger}(M_0+M_1j)$, e.g., (M_0+M_1j) is q-EP.

Thus (i) holds.

$(ii)\Leftrightarrow(iii)$ By corollary 8 in [10]

$\begin{pmatrix}A_0+A_1j & 0\\ C_0+C_1j & (M_0+M_1j)/(A_0+A_1j)\end{pmatrix}$ is q-EP, iff A_0+A_1j and

$((M_0+M_1j)/(A_0+A_1j))$ are q-EP,

Further $N(A_0+A_1j)\subseteq N(C_0+C_1j)$ and $N((M_0+M_1j)/(A_0+A_1j)^{*})\subseteq N(C_0+C_1j)^{*}$

$\begin{pmatrix}(A_0+A_1j) & (B_0+B_1j)\\ 0 & (M_0+M_1j)/(A_0+A_1j)\end{pmatrix}$

is q-EP iff A and $(M_0+M_1j)/(A_0+A_1j)$ are q-EP, further $N(A_0+A_1j)^{*}\subseteq N(B_0+B_1j)^{*}$ and $N((M_0+M_1j)/(A_0+A_1j))\subseteq N(B_0+B_1j)$. This proves the equivalence of (ii) and (iii).

The proof is complete.

$$(M_0+M_1j)=\begin{bmatrix}1&1&1&j\\1&1&1&-j\\1&1&1&1\\j&0&j&0\end{bmatrix}+\begin{bmatrix}0&0&0&-1\\0&0&0&1\\0&0&0&0\\-1&0&-1&0\end{bmatrix}j=\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&1&1\\0&0&0&0\end{bmatrix}$$

Theorem 1.3 :

Let (M_0+M_1j) be a matrix of the form(2) with $N(A_0+A_1j)^{*}\subseteq N(B_0+B_1j)^{*}$ and $N((M_0+M_1j)/(A_0+A_1j))^{*}\subseteq N(C_0+C_1j)^{*}$, then the following are equivalent.

i. (M_0+M_1j) is an q-EP matrix

ii. (A_0+A_1j) and $((M_0+M_1j)/(A_0+A_1j))$ are q-EP, further $N(A_0+A_1j)\subseteq N(C_0+A_1j)$ and $N((M_0+M_1j)/(A_0+A_1j))\subseteq N(B_0+B_1j)$;

iii. Both the matrices
$$\begin{pmatrix}(A_0+A_1j) & 0\\ (C_0+C_1j) & (M_0+M_1j)/(A_0+A_1j)\end{pmatrix} \text{ and } \begin{pmatrix}(A_0+A_1j) & (B_0+B_1j)\\ 0 & (M_0+M_1j)/(A_0+A_1j)\end{pmatrix}$$
are q-EP.

Proof :

Theorem 1.3 follows immediately from theorem 1.2 and from the fact that (M_0+M_1j) is q-EP iff $(M_0+M_1j)^*$ is q-EP iff $(M_0+M_1j)^*$ is q-EP.

In this special case when $(B_0+B_1j)=(C_0+C_1j)^*$ we get the following

Corollary 1.4 :

Let $(M_0+M_1j)=\begin{pmatrix}(A_0+A_1j) & (C_0+C_1j)^*\\ (C_0+C_1j) & (D_0+D_1j)\end{pmatrix}$ with $N(A_0+A_1j)\subseteq N(C_0+C_1j)$ and $N((M_0+M_1j)/(A_0+A_1j))\subseteq N(C_0+C_1j)^*$, then the following are equivalent.

i. (M_0+M_1j) is an q-EP matrix

ii. (A_0+A_1j) and $((M_0+M_1j)/(A_0+A_1j))$ are q-EP matrices.

iii. the matrix $\begin{pmatrix}(A_0+A_1j) & 0\\ (C_0+C_1j) & (M_0+M_1j)/(A_0+A_1j)\end{pmatrix}$ is q-EP.

Remark 1.5 :

The conditions that taken on $M=M_0+M_1j$ in the previous theorems are essential. This is illustrated in the following example. Let

$$M=\begin{bmatrix}1&1&1&1+j\\1&1&1-j&1\\1&1-j&1&1\\1+j&1&1&0\end{bmatrix}+\begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{bmatrix}j=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&0\end{bmatrix}$$

M is symmetric and

$$(B_0+B_1j)=(C_0+C_1j)=\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

$$((M_0+M_1j)/(A_0+A_1j))=(D_0+D_1j)-(C_0+C_1j)(A_0+A_1j)^{\dagger}(B_0+B_1j)=\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

Clearly (A_0+A_1j) and $((M_0+M_1j)/(A_0+A_1j))$ are q-EP, $N(A_0+A_1j)\subseteq N(C_0+C_1j)$ and $N(A_0+A_1j)^*\subseteq N(B_0+B_1j)^*$, but $N((M_0+M_1j)/(A_0+A_1j))\subseteq N(B_0+B_1j)$ and $N((M_0+M_1j)/(A_0+A_1j))^*\not\subset N(C_0+C_1j)^*$, further

$$\begin{pmatrix}(A_0+A_1j) & 0\\ (C_0+C_1j) & (M_0+M_1j)/(A_0+A_1j)\end{pmatrix} \text{ and } \begin{pmatrix}(A_0+A_1j) & (B_0+B_1j)\\ 0 & (M_0+M_1j)/(A_0+A_1j)\end{pmatrix}$$

or not q-EP. Thus theorem 1.2 and 1.3 as well as corollary 1.4 fail.

Remark 1.6 :

We conclude from Theorem 1.2 and Theorem 1.3 that for an q-EP matrix M of the form equation (2) the following are equivalent

$$N(A_0+A_1j)\subseteq N(C_0+C_1j),\ N((M_0+M_1j)/(A_0+A_1j))\subseteq N(B_0+B_1j) \tag{4}$$

$$N(A_0+A_1j)^*\subseteq N(B_0+B_1j)^*,\ N((M_0+M_1j)/(A_0+A_1j))^*\subseteq N(C_0+C_1j)^* \tag{5}$$

However this fails if we omit the condition that (M_0+M_1j) is q-EP. For example Let

$$(M_0+M_1j)=\begin{bmatrix}1&1&1&j\\1&1&1&-j\\1&1&1&1\\j&0&j&0\end{bmatrix}+\begin{bmatrix}0&0&0&-1\\0&0&0&1\\0&0&0&0\\-1&0&-1&0\end{bmatrix}j=\begin{bmatrix}1&1&1&0\\1&1&1&0\\1&1&1&1\\0&0&0&0\end{bmatrix}$$

(M_0+M_1j) is not q-EP. Here

$$(A_0+A_1j)=\begin{pmatrix}1&1\\1&1\end{pmatrix},\ (B_0+B_1j)=(C_0+C_1j)^*=\begin{pmatrix}1&0\\1&0\end{pmatrix}$$

(A_0+A_1j) is q-EP, $N(A_0+A_1j)\subseteq N(C_0+C_1j)$ and $N(A_0+A_1j)^*\subseteq N(B_0+B_1j)^*$.

Hence $((M_0+M_1j)/(A_0+A_1j))$ is independent of the choice of $(A_0+A_1j)^-$ and so

$$((M_0+M_1j)/(A_0+A_1j))=(D_0+D_1j)-(C_0+C_1j)(A_0+A_1j)^\dagger(B_0+B_1j)=\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$((M_0+M_1j)/(A_0+A_1j))$ is not q-EP, $N((M_0+M_1j)/(A_0+A_1j))^*\subseteq N(C_0+C_1j)^*$, but $N(A_0+A_1j)\subseteq N(B_0+B_1j)$, Thus equation(5) holds, while equation(5) fails.

Remark 1.7 :

It has been proved is[2] that for any matrix A its Moore-Penrose inverse. $(M_0+M_1j)^\dagger$ is given by the formula equation(3) iff both equation (4) and equation (5) holds. However it is clear by the previous remark 1.6 that for an q-EP matrix formula (3) gives $(M_0+M_1j)^{\dagger}$ iff either (4) or (5) holds.

Theorem 1.8 :

Let (M_0+M_1j) be of the form equ(2) with $rk(M_0+M_1j)=rk(A_0+A_1j)=r$. Then (M_0+M_1j) is an q-EP$_r$, matrix if and only if A is q-EP, and

$(C_0+C_1j)(A_0+A_1j)^{\dagger}=((A_0+A_1j)^{\dagger}(B_0+B_1j))^*$.

Proof :

Since $rk(M_0+M_1j)=rk(A_0+A_1j)=r$, we have by reason of the corollary of theorem (1) in [3] that,

$N(A_0+A_1j)\subseteq N(C_0+C_1j)$, $N(A_0+A_1j)^*\subseteq N(B_0+B_1j)^*$ and

$(M_0+M_1j)/(A_0+A_1j)=(D_0+D_1j)-(C_0+C_1j)(A_0+A_1j)^{\dagger}(B_0+B_1j)=0$. According to Theorem 1.1 these relation are equivalent $(C_0+C_1j)=(C_0+C_1j)(A_0+A_1j)^{\dagger}(A_0+A_1j)$,

$(B_0+B_1j)=(A_0+A_1j)(A_0+A_1j)^{\dagger}(B_0+B_1j)$ and $(D_0+D_1j)=(C_0+C_1j)(A_0+A_1j)^{\dagger}(B_0+B_1j)$. Let us consider the matrices

$$P=\begin{pmatrix}(I_0+I_1j) & 0\\ (C_0+C_1j)(A_0+A_1j)^{\dagger} & (I_0+I_1j)\end{pmatrix},\ Q=\begin{pmatrix}(I_0+I_1j) & (A_0+A_1j)^{\dagger}(B_0+B_1j)\\ 0 & (I_0+I_1j)\end{pmatrix},$$

$$L=\begin{pmatrix}(A_0+A_1j) & 0\\ 0 & 0\end{pmatrix}.$$

P and Q are non-singular and by assumption

$(C_0+C_1j)(A_0+A_1j)^{\dagger}=((A_0+A_1j)^{\dagger}(B_0+B_1j))^*$ it holds $P=(Q_0+Q_1j)^*$. Therefore (M_0+M_1j) can be factorized as $M=(P_0+P_1j)(L_0+L_1j)(P_0+P_1j)^*$. Since (A_0+A_1j) is q-EP$_r$ consequently (L_0+L_1j) is as well q-EP$_r$.

Hence $N(L_0+L_1j)=N(L_0+L_1j)$ and so we have according to Lemma 3 of paper[1] that
$N(M_0+M_1j)=N(P_0+P_1j)(L_0+L_1j)(P_0+P_1j)^*=N((P_0+P_1j)(L_0+L_1j)^*(P_0+P_1j)^*)=N(M_0+M_1j)^*)$
This shows that is q-EP$_r$.

Conversely, Let us assume that (M_0+M_1j) is q-EP$_r$. Since

$(M_0+M_1j)=(P_0+P_1j)(L_0+L_1j)(Q_0+Q_1j)$, one choice of $(A_0+A_1j)^-$ is

$$(M_0+M_1 j)^- = (Q_0+Q_1 j)^{-1}\begin{pmatrix}(A_0+A_1 j)^\dagger & 0\\ 0 & 0\end{pmatrix}(P_0+P_1 j)^{-1}$$

We know that $N(M_0+M_1 j)=N((M_0+M_1)^*)$, therefore by

Lemma 1.1 $(M_0+M_1 j)^* = (M_0+M_1 j)^*(M_0+M_1 j)^-(M_0+M_1 j)$ holds, e.g

$$(M_0+M_1 j)^* = \begin{pmatrix}(A_0+A_1 j)^* & (C_0+C_1 j)^*\\ (B_0+B_1 j)^* & (D_0+D_1 j)^*\end{pmatrix}$$

$$= \begin{pmatrix}(A_0+A_1 j)^* & (C_0+C_1 j)^*\\ (B_0+B_1 j)^* & (D_0+D_1 j)^*\end{pmatrix}\begin{pmatrix}(A_0+A_1 j)^\dagger(A_0+A_1 j) & (A_0+A_1 j)^\dagger(B_0+B_1 j)\\ 0 & 0\end{pmatrix}$$

or equivalently, $(A_0+A_1 j)^* = (A_0+A_1 j)^*(A_0+A_1 j)^\dagger(A_0+A_1 j)$ and

$(C_0+C_1 j)^* = (C_0+C_1 j)^*(A_0+A_1 j)^\dagger(B_0+B_1 j)$, $(A_0+A_1 j)^* = (A_0+A_1 j)^*(A_0+A_1 j)^\dagger(A_0+A_1 j)$

it follows $N(A_0+A_1 j)^* = N(A_0+A_1 j)$, i.e., A is q-EP$_r$ and therefore

$(A_0+A_1 j)(A_0+A_1 j)^\dagger = (A_0+A_1 j)^\dagger(A_0+A_1 j)$ taking into account

$(C_0+C_1 j)^* = (C_0+C_1 j)^*(A_0+A_1 j)^\dagger(B_0+B_1 j)$, we have

$$(C_0+C_1 j)(A_0+A_1 j)^\dagger = (B_0+B_1 j)^*((A_0+A_1 j)^\dagger)^*((A_0+A_1 j)^\dagger(A_0+A_1 j))$$

$$= (B_0+B_1 j)^*((A_0+A_1 j)^\dagger(A_0+A_1 j)(A_0+A_1 j)^\dagger)^*$$

$$= (B_0+B_1 j)^*((A_0+A_1 j)^\dagger)^* = ((A_0+A_1 j)^\dagger(B_0+B_1 j))^*$$

The theorem is proved.

Corollary 1.9 :

Let $(M_0+M_1 j)$ of the form (2) with $(A_0+A_1 j)$ non-singular matrix and $rk(M_0+M_1 j)=rk(A_0+A_1 j)$. Then M is q-EP if and only if $(C_0+C_1 j)(A_0+A_1 j)^\dagger = ((A_0+A_1 j)^\dagger(B_0+B_1 j))^*$.

Corollary 1.10 :

Let M= $(M_0+M_1 j)$ be an $n\times n$ matrix f rank r. Then $(M_0+M_1 j)$ is q-EP$_r$ if and only if every principal sub matrix of rank r is q-EP$_r$.

Proof :

Suppose M= $(M_0 + M_1 j)$ is an q-EP_r matrix. Let $(A_0 + A_1 j)$ be any principal submatrix of $(M_0 + M_1 j)$ such that $rk(M_0 + M_1 j) = rk(A_0 + A_1 j) = r$. Then there exists a permutation matrix such that

$$\tilde{M} = PMP^T = \begin{pmatrix} (A_0 + A_1 j) & (B_0 + B_1 j) \\ (C_0 + C_1 j) & (D_0 + D_1 j) \end{pmatrix} \text{ and } rk(A_0 + A_1 j) = r$$

According to Lemma (3) in[1], M is q-EP_r. Now, we conclude from theorem (1.3) that A q-EP_r as well. Since A was arbitrary, it follows that very principal submatrix of rank r is q-EP_r . The converse is obvious.

Ramark 1.11 :

Theorem 1.8 fails if we relax the condition on rank of M= $(M_0 + M_1 j)$.

2. Application :

We give conditions under which a partitioned matrix is decomposed into complementary summands of q-EP matrices. M_1 and M_2 are called complementary summand of $(M_0 + M_1 j)$ if $M = M_1 + M_2$ and $rk(M) = rk(M_1) + rk(M_2)$.

Theorem 2.1

Let $(M_0 + M_1 j)$ of the form (2) with $rk(M_0 + M_1 j) = rk(A_0 + A_1 j) = rk((M_0 + M_1 j)/(A_0 + A_1 j)$, where $((M_0 + M_1 j)/(A_0 + A_1 j)) = (D_0 + D_1 j) - (C_0 + C_1 j)(A_0 + A_1 j)^{\dagger}(B_0 + B_1 j)$. If $(A_0 + A_1 j)$ and $((M_0 + M_1 j)/(A_0 + A_1 j))$ are q-EP matrices such that $(C_0 + C_1 j)(A_0 + A_1 j)^{\dagger} = ((A_0 + A_1 j) + (B_0 + B_1 j))^*$ and $B((M_0 + M_1 j)/(A_0 + A_1 j))^{\dagger} = (((M_0 + M_1 j)/(A_0 + A_1 j))^{\dagger}(C_0 + C_1 j)^*)$ then $(M_0 + M_1 j)$ can be decomposed into complementary summands of q-EP matrices.

Proof :

Let us consider the matrices

$$M_1 = \begin{pmatrix} (A_0 + A_1 j) & (A_0 + A_1 j)(A_0 + A_1 j)^{\dagger}(B_0 + B_1 j) \\ (C_0 + C_1 j)(A_0 + A_1 j)^{\dagger}(A_0 + A_1 j) & (C_0 + C_1 j)(A_0 + A_1 j)^{\dagger}(B_0 + B_1 j) \end{pmatrix} \text{ and}$$

$$M_2 = \begin{pmatrix} 0 & ((I_0 + I_1 j) - (A_0 + A_1 j)(A_0 + A_1 j)^{\dagger})(B_0 + B_1 j) \\ (C_0 + C_1 j)((I_0 + I_1 j) - (A_0 + A_1 j)^{\dagger}(A_0 + A_1 j)) & (M_0 + M_1 j)/(A_0 + A_1 j) \end{pmatrix}$$

Taking into account that $N(A_0 + A_1 j) \subseteq N((C_0 + C_1 j)(A_0 + A_1 j)^{\dagger}(A_0 + A_1 j))$, $N((A_0 + A_1 j)^*) \subseteq N((A_0 + A_1 j)(A_0 + A_1 j)^{\dagger}(B_0 + B_1 j))^*$ and

$$M_1/A=(C_0+C_1j)(A_0+A_1j)^{\dagger}(B_0+B_1j)-(((C_0+C_1j)(A_0+A_1j)^{\dagger}(A_0+A_1j))(A_0+A_1j)-((A_0+A_1j)(A_0+A_1j)^{\dagger}(B_0+B_1j)=(C_0+C_1j)(A_0+A_1j)$$

we obtain by the corollary after Theorem 1 in[5], that $rk(M_1)=rk(A_0+A_1j)$.

Since (A_0+A_1j) is q-EP and $((C_0+C_1j)(A_0+A_1j)^{\dagger}(A_0+A_1j))(A_0+A_1j)^{\dagger}=(C_0+C_1j)(A_0+A_1j)^{\dagger}$

$$=((A_0+A_1j)^{\dagger}(B_0+B_1j))^{*}=((A_0+A_1j)^{\dagger}(A_0+A_1j)(A_0+A_1j)^{\dagger}(B_0+B_1j))^{*}$$

We have from Theorem 1.8 that M_1 is q-EP. Since

$rk(M)=rk(A_0+A_1j)+rk((M_0+M_1j)/(A_0+A_1j))$, Theorem 1. Of paper[5]

gives $N((M_0+M_1j)/(A_0+A_1j))\subseteq N((I_0+I_1j)-(A_0+A_1j)(A_0+A_1j)^{\dagger})(B_0+B_1j)$,

$N((M_0+M_1j)/(A_0+A_1j))\subseteq N(((I_0+I_1j)-(A_0+A_1j)^{\dagger})(C_0+C_1j))^{*}$ and

$$((I_0+I_1j)-(A_0+A_1j)(A_0+A_1j)^{\dagger})M((M_0+M_1j)/(A_0+A_1j))^{\dagger}(C_0+C_1j)((I_0+I_1j)-(A_0+A_1j)^{\dagger}(A_0+A_1j))=0$$

Thus by the corollary of the just applied Theorem 1.1 in[5], we

have $rk(M_2)=rk(M_0+M_1j)/(A_0+A_1j))$.

Further, using $(A_0+A_1j)(A_0+A_1j)^{\dagger}=(A_0+A_1j)^{\dagger}(A_0+A_1j)$, we obtain

$$((I_0+I_1j)-(A_0+A_1j)(A_0+A_1j)^{\dagger})(B_0+B_1j)(M_0+M_1j))/(A_0+A_1j))$$

$$=((I_0+I_1j)-(A_0+A_1j)(A_0+A_1j)^{\dagger})(((M_0+M_1j)/(A_0+A_1j))^{\dagger}(C_0+C_1j))^{*}$$

$$=(((M_0+M_1j)/(A_0+A_1j))^{\dagger}(C_0+C_1j)((I_0+I_1j)-(A_0+A_1j)(A_0+A_1j)))^{*}$$

$$=(((M_0+M_1j)/(A_0+A_1j))^{\dagger}(C_0+C_1j)((I_0+I_1j)-(A_0+A_1j)^{\dagger}(A_0+A_1j))^{*}$$

Thus by Theorem 1.8 M_2 is also q-EP. Clearly $M=M_1+M_2$, where both M_1 and M_2 are q-EP matrices and $rk(M_0+M_1j)=rk(A)+rk((M_0+M_1j)/(A0+A_1j))=rk(M_1)+rk(M_2)$.

Hence M_1 and M_2 are complementary summands of q-EP matrices.

References

1. T.S.Baskett and I.J.Katz, Theorems on products of EP, matrices, Linear Algebra and its Appl. 2 (1969), 87-103. MR 40: 4280
2. F.Burns, D.Carlson, E.Haynsworth and TH.Markham, Generalized inverse formulas using the Schur Complement, SIAM J .Appl. Math. *26*, 254-259. MR *48*, 8519 (1974).

3. D.Carlson, E.Haynsworth and TH.Markham, A generalization of the Schur complement by means of the Moore-Penrose inverse, SIAM *J. Appl. Math. 26*, 169-175. MR 50: 344 (1974)
4. Gunasekaran.K and Sridevi.S., On Range Quaternion Hermitian Matrices; Inter; J; Math., Archieve-6(8), 159-163 (2015).
5. Gunasekaran. K. and Sridevi.S., On Sums of Range Quaternion Hermitian Matrices; Inter; J. Modern Engineering Research -.5, ISS.11. 44 – 49 (2015).
6. Gunasekaran. K. and Sridevi. S., On Product of Range Quaternion Hermitian Matrices; Inter., Journal of pure algebra – *6(6),* 318-321 (2016).
7. Gunasekaran. K. and Sridevi. S., Generalized inverses, Group inverses and Revers order law of range quaternion Hermitian matrices., IOSR-JM, Vol.,12, Iss.4 ver.II (Jul – Aug 2016) , 51- 55.
8. Gunasekaran. K. and Sridevi. S., On Bicomplex Representation Methods and Application of Range Quaternion Hermitian Matrices (q-Ep)., IJMTT- Vol. *48,* No. 4 August, 250 – 259 (2017).
9. Gunasekaran. K. and Sridevi. S., On Schur Complements in Range Quaternion Hermitian Matrices., IJMAA-Vol. *5,* Issue 4-F, 985-991 (2017).
10. M.H. Pearl, On generalized inverses of matrices, Proc. Cambridge Phil. Soc. *62*, 673-677. MR 33: 5650 (1966).
11. C.D. Meyer, Generalized inverses of block triangular matrices, SIAM J. Appl. Math. *19,* 741- 750. Mr 42: 7676 (1970).
12. R. Penrose, A Generalized inverse for matrices, Proc. Cambridge Philos. Soc. *51*, 406-415. MR 16, 1082 (1955).
13. C.R. Rao and S.K. Mitra, Generalized inverse of matrices and its applications, Wiley, New York, .MR 49: 2780 (1971).
14. P. Robert, On the group inverse of a linear transformation, J. Math. Anal.Appl. 22 , 658-669. MR 37:5232(1968).
15. H. Schwerdtfeger, Introduction to linear algebra and the theory of matrices, Noordhoff, Groningen, 1950. Mr12, 470