ISSN 2231-346X (Print)
JUSPS-A Vol. 30(11), 395-400 (2018). Periodicity-Monthly
Section A
ISSN 2319-8044 (Online)

JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES
An International Open Free Access Peer Reviewed Research Journal of Mathematics
website:- www.ultrascientist.org
Estd. 1989

On Pre-generalized c*-homeomorphisms in topological spaces

[1]S. MALATHI and [2]S. NITHYANANTHA JOTHI

[1]Assistant Professor, Department of Mathematics, Wavoo Wajeeha Women's College of Arts and Science, Kayalpatnam, Tiruchendur-628215, Tamilnadu (India)
[2]Assistant Professor, Department of Mathematics, Aditanar College of Arts and Science, Tiruchendur-628215, Tamilnadu (India)
Corresponding of Author Email :- [1]malathis2795@gmail.com, [2]nithananthajothi@gmail.com
http://dx.doi.org/10.22147/jusps-A/301101

Acceptance Date 10th October, 2018, Online Publication Date 2nd November, 2018

Abstract

The aim of this paper is to introduce the notion of pre-generalized c*-homeomorphisms in topological spaces and study their basic properties.
Key words: pgc*-open maps, pgc*-continuous functions and pgc*- homeomorphisms.

1. Introduction

Norman Levine introduced the concept of semi-continuous function in 1963. In 1980, Jain introduced totally continuous functions. In 2011, S.S. Benchalli and Umadevi I Neeli introduced the concept of semi-totally continuous functions in topological spaces. H. Maki *et. al.* introduced and investigated generalized homeomorphisms and gc-homeomorphisms. R. Devi *et. al.* introduced and studied semi-generalized homeomorphisms and generalized semi-homeomorphisms. In this paper, we introduce pre-generalized c*-homeomorphisms in topological spaces and study their basic properties.

Section 2 deals with the preliminary concepts. In section 3, pre-generalized c*- homeomorphisms in topological spaces are introduced and their basic properties are studied.

2. Preliminaries :

Throughout this paper X denotes a topological space on which no separation axiom is assumed. For any subset A of X, cl(A) denotes the closure of A, int(A) denotes the interior of A, pcl(A) denotes the pre-closure of A and bcl(A) denotes the b-closure of A. Further X\A denotes the complement of A in X. The following definitions are very useful in the subsequent sections.

Definition: 2.1 A subset A of a topological space X is called

i. a semi-open set[4] if $A\subseteq cl(int(A))$ and a semi-closed set if $int(cl(A))\subseteq A$.
ii. a pre-open set[12] if $A\subseteq int(cl(A))$ and a pre-closed set if $cl(int(A))\subseteq A$.

Definition: 2.2[5] A subset A of a topological space X is said to be a c*-open set if $int(cl(A))\subseteq A\subseteq cl(int(A))$.

Definition: 2.3 A subset A of a topological space X is called

i. a generalized pre-regular closed set (briefly, gpr-closed)[2] if $pcl(A)\subseteq H$ whenever $A\subseteq H$ and H is regular-open in X.
ii. a weakly closed set (briefly, w-closed)[15] (equivalently, $\hat{g}$-closed [16]) if $cl(A)\subseteq H$ whenever $A\subseteq H$ and H is semi-open in X.

The complements of the above mentioned closed sets are their respectively open sets.

Definition: 2.4[5] A subset A of a topological space X is said to be a generalized c*-closed set (briefly, gc*-closed set) if $cl(A)\subseteq H$ whenever $A\subseteq H$ and H is c*-open. The complement of the gc*-closed set is gc*-open [6].

Definition: 2.5[8] A subset A of a topological space X is said to be a pre-generalized c*-closed set (briefly, pgc*-closed set) if $pcl(A)\subseteq H$ whenever $A\subseteq H$ and H is c*-open. The complement of the pgc*-closed set is pgc*-open[9].

Definition: 2.6 A function $f : X \rightarrow Y$ is called

i. totally-continuous[3] if the inverse image of every open subset of Y is clopen in X.
ii. strongly-continuous[13] if the inverse image of every subset of Y is clopen subset of X.
iii. semi-totally continuous[1] if the inverse image of every semi- open subset of Y is clopen in X.
iv. gpr-continuous[2] if inverse image of every closed subset of Y is gpr-closed in X.
v. w-continuous[14] (equivalently, $\hat{g}$-continuous[16]) if inverse image of every closed subset of Y is w-closed in X.

Definition: 2.7[16] A function $f : X \rightarrow Y$ is said to be a $\hat{g}$-open map if f(U) is $\hat{g}$-open in Y for every open set U of X.

Definition: 2.8[6] A function $f : X \rightarrow Y$ is said to be a generalized c*-open (briefly, gc*-open) map if f(U) is gc*-open in Y for every open set U of X.

Definition: 2.9[9] A function $f : X\rightarrow Y$ is said to be a pre-generalized c*-open (briefly, pgc*-open) map if f(U) is pgc*-open in Y for every open set U of X.

Definition: 2.10[7] Let X and Y be two topological spaces. A function $f : X \rightarrow Y$ is called a generalized c*-continuous (briefly, gc*-continuous) function if $f^{-1}(V)$ is gc*-closed in X for every closed set V of Y .

Definition: 2.11[10] Let X and Y be two topological spaces. A function $f : X \rightarrow Y$ is called a pre-generalized c*-continuous (briefly, pgc*-continuous) function if $f^{-1}(V)$ is pgc*-closed in X for every closed set V of Y.

Definition: 2.12[16] A bijective function $f : X \rightarrow Y$ is called a $\hat{g}$-homeomorphism if f is both $\hat{g}$-continuous and $\hat{g}$-open.

Definition: 2.13[11] A bijective function $f : X\rightarrow Y$ is said to be generalized c*-homeomorphism (briefly, gc*-homeomorphism) if f is both gc*-continuous and gc*-open map.

3. Pre-generalized c-homeomorphisms :*

In this section, we introduce pre-generalized c*-homeomorphisms and study their basic properties.

Definition: 3.1 A bijective function f : X→Y is said to be pre-generalized c*-homeomorphism (briefly, pgc*-homeomorphism) if f is both pgc*-continuous and pgc*-open map.

Example: 3.2 Let X={a,b,c} and Y={1,2,3}. Then, clearly τ={ϕ,{b},{c},{b,c},X} is a topology on X and σ={ϕ, {1},Y} is a topology on Y. Define f : X → Y by f(a)=1, f(b)=3, f(c)=2. Then f is both pgc*-continuous and pgc*-open map. Therefore, f is a pgc*-homeomorphism.

Proposition: 3.3 Let X,Y be topological spaces. Then every homeomorphism is a pgc*-homeomorphism.

Proof: Let f : X→Y be a homeomorphism. Then f is both continuous and open map. By Proposition 3.4[10], f is pgc*-continuous and by Proposition 4.4[9], f is a pgc*-open map. Therefore, f is pgc*-homeomorphism.

The converse of Proposition 3.3 need not be true which can be verified from the following example.

Example: 3.4 In Example 3.2, the image of the open set {b} in X is {3}, which is not open in Y. Therefore, f is not homeomorphism.

Proposition: 3.5 Let X be a topological space. Then every $\hat{g}$-homeomorphism is a pgc*-homeomorphism.

Proof: Let f : X→ Y be a $\hat{g}$-homeomorphism. Then f is both $\hat{g}$-continuous and $\hat{g}$-open map. By Proposition 3.4 [10], f is pgc*-continuous. Also, by Proposition 4.6 [9], f is a pgc*-open map. Therefore, f is pgc*-homeomorphism.

The converse of Proposition 3.5 need not be true as seen from the following example.

Example: 3.6 In Example 3.2, the function f : X→Y is a pgc*-homeomorphism. But the inverse image of the closed set {2,3} in Y under f is {b,c}, which is not a $\hat{g}$-closed set in X. Therefore, f is not a $\hat{g}$-continuous function. Hence f is not a $\hat{g}$-homeomorphism.

Proposition: 3.7 Let X be a topological space. Then every gc*-homeomorphism is a pgc*-homeomorphism.

Proof: Let f : X → Y be a gc*-homeomorphism. Then f is both gc*-continuous and gc*-open map. By Proposition 3.4 [10], f is pgc*-continuous. Since every gc*-open map is pgc*-open map, we have f is a pgc*-open map. Therefore, f is a pgc*-homeomorphism.

The following example shows that the converse the Proposition 3.7 need not be true.

Example: 3.8 Let X={a,b,c,d,e} and Y={1,2,3,4,5}. Then, clearly τ={ϕ,{a,b},{c,d},{a,b,c,d}, X} is a topology on X and σ={ϕ, {1},{2},{1,2},{1,2,3},{1,2,3,4}{1,2,3,5},Y} is a topology on Y. Define f : X→Y by f(a)=1, f(b)=2, f(c)=3, f(d)=4, f(e)=5. Then f is a pgc*-homeomorphism. But f is not a gc*-homeomorphism, since the inverse image of the closed set {4} in Y under f is {d}, which is not a gc*-closed set in X.

The composition of two pgc*-homeomorphisms need not be a pgc*-homeomorphism. For example, let X={a,b,c}, Y={1,2,3} and Z={p,q,r}. Then, clearly τ={ϕ,{b},{c},{b,c},X} is a topology on X, σ={ϕ,{1},Y} is a topology on Y and η={ϕ,{p},{p,q},Z} is a topology on Z. Define f : X →Y by f(a)=1, f(b)=3, f(c)=2 and define g : Y→Z by g(1)=q, g(2)=p, g(3)=r. Then f and g are pgc*-homeomorphisms. Consider the closed set {r} in Z. Then $(g\circ f)^{-1}(\{r\})=f^{-1}(g^{-1}(\{r\}))= f^{-1}(\{3\})=\{b\}$, which is not a pgc*-closed set in X. Therefore, g∘f is not a pgc*-homeomorphism.

Proposition: 3.9 Let X,Y,Z be topological spaces. If f : X→Y and g : Y → Z are homeomorphisms, then g∘f : X→Z is a pgc*-homeomorphism.

Proof: Assume that f : X→Y and g : Y→Z are homeomorphisms. Then f and g are both continuous and open maps. By Proposition 3.10 [10], g∘f is a pgc*-continuous function. Also, by Proposition 4.9 [9], g∘f is a pgc*-open map. Hence g∘f is a pgc*-homeomorphism.

Proposition: 3.10 Let X,Y be topological spaces. If f : X→Y is strongly continuous and image of every subset of X is a clopen subset of Y, then f is pgc*-homeomorphism.

Proof: Let f : X→Y be a strongly continuous function. Then by Proposition 3.4 [10], f is a pgc*-continuous function. Now, let U be a open set in X. By our assumption, f(U) is a clopen in Y. By Proposition 3.7 [6], f(U) is gc*-open in Y. This implies, f(U) is pgc*-open in Y. Therefore, f is a pgc*-open map. Hence f is a pgc*-homeomorphism.

Proposition: 3.11 Let X,Y be topological spaces. If f : X→Y is a semi-totally continuous function and image of every semi-open subset of X is clopen in Y, then f is pgc*-homeomorphism.

Proof: Let f : X→Y be a semi-totally continuous function. Then by Proposition 3.4 [10], f is a pgc*-continuous function. Now, let U be a open set in X. Then U is semi-open in X. By our assumption, f(U) is a clopen in Y. By Proposition 3.7 [6], f(U) is gc*-open in Y. This implies, f(U) is pgc*-open in Y. Therefore, f is a pgc*-open map. Hence f is a pgc*-homeomorphism.

Proposition: 3.12 Let X,Y be topological spaces. If f : X→Y is a totally continuous function and image of every open subset of X is clopen in Y, then f is pgc*-homeomorphism.

Proof: Let f : X→Y be a totally continuous function. Then by Proposition 3.4 [10], f is a pgc*-continuous function. Now, let U be a open set in X. By our assumption, f(U) is a clopen in Y. By Proposition 3.7 [6], f(U) is gc*-open in Y. This implies, f(U) is pgc*-open in Y. Therefore, f is a pgc*-open map. Hence f is a pgc*-homeomorphism.

Proposition: 3.13 Let X,Y be topological spaces. If f : X→Y is a pgc*-homeomorphism, then f is gpr-continuous and image of every closed subset of X is gpr-closed in Y.

Proof: Assume that f is a pgc*-homeomorphism. Then f is both pgc*-continuous and pgc*-open map. Then by Proposition 3.6 [10], f is gpr-continuous. Now, let V be a closed set in X. Since f is a pgc*-open map, by Proposition 4.3 [9], f(V) is a pgc*-closed set in Y. Therefore, by Proposition 3.15 [8], f(V) is gpr-closed in X. Hence the proof.

Proposition: 3.14 Let X,Y be a topological space. A bijective function f : X→Y is a pgc*-homeomorphism if and only if f is pgc*-continuous and f^{-1} : Y →X is pgc*-continuous.

Proof: Assume that f is a pgc*-homeomorphism. Then f is pgc*-continuous and pgc*-open map. By Proposition 3.8 [10], f^{-1} : Y→ X is a pgc*-continuous function. Conversely, assume that f is pgc*-continuous and f^{-1} is pgc*-continuous. Then by Proposition 3.8 [10], f : X →Y is a pgc*-open map. Hence f is a pgc*-homeomorphism.

Proposition: 3.15 Let X,Y and Z be topological spaces. If f : X →Y is pgc*-homeomorphism and g : Y→ Z is totally-continuous and if g(U) is pgc*-open for every pgc*-open set U in Y, then g∘f : X →Z is pgc*-homeomorphism.

Proof: Let V be an open set in Z. Then $g^{-1}(V)$ is clopen in Y. This implies, $g^{-1}(V)$ is open in Y. Since f is pgc*-continuous, we have $f^{-1}(g^{-1}(V))$ is pgc*-open. That is, $(g \circ f)^{-1}(V)$ is pgc*-open in X. Therefore, g∘f is pgc*-continuous. Let U be an open set in X. Then f(U) is pgc*-open in Y. This implies, g(f(U)) is pgc*-open in Z. That is, (g∘f)(U) is pgc*-open in Z. Therefore, g∘f is pgc*-open map. Hence g∘f is pgc*-homeomorphism.

Proposition: 3.16 Let X,Y and Z be topological spaces. If f : X→Y is pgc*-homeomorphism and g : Y→Z is semi-totally continuous and if g(U) is pgc*-open for every pgc*-open set U in Y, then g∘f : X→Z is pgc*-homeomorphism.

Proof: Let V be an open set in Z. Then V is semi-open in Z. This implies, $g^{-1}(V)$ is clopen in Y. Since f is pgc*-continuous, we have $f^{-1}(g^{-1}(V))$ is pgc*-open. That is, $(g\circ f)^{-1}(V)$ is pgc*-open in X. Therefore, g∘f is pgc*-continuous. Let U be an open set in X. Then f(U) is pgc*-open in Y. This implies, g(f(U)) is pgc*-open in Z. That is, (g∘f)(U) is pgc*-open in Z. Therefore, g∘f is pgc*-open map. Hence g∘f is pgc*-homeomorphism.

Proposition: 3.17 Let X,Y and Z be topological spaces. If f : X→Y is both open and strongly-continuous and g : Y→Z is pgc*-homeomorphism, then g∘f : X→Z is pgc*-homeomorphism.

Proof: Let V be an open set in Z. Then $g^{-1}(V)$ is pgc*-open in Y. Since f is strongly-continuous, we have $f^{-1}(g^{-1}(V))$ is clopen in X. That is, $(g\circ f)^{-1}(V)$ is pgc*-open in X. Therefore, g∘f is pgc*-continuous. Let U be an open set in X. Then f(U) is open in Y. This implies, g(f(U)) is pgc*-open in Z. That is, (g∘f)(U) is pgc*-open in Z. Therefore, g∘f is pgc*-open map. Hence g∘f is pgc*-homeomorphism.

Conclusion

In this paper we have introduced pgc*-homeomorphisms in topological spaces. Also, we have studied the relationship between pgc*-homeomorphism and other continuous functions already exist.

References

1. S.S. Benchalli and U. I Neeli, Semi-totally Continuous function in topological spaces, Inter. Math. Forum, 6, *10*, 479-492 (2011).
2. Y. Gnanambal, On generalized pre regular closed sets in topological spaces, *Indian J.Pure Appl. Math., 28*, 351-360 (1997).
3. R.C. Jain, The role of regularly open sets in general topological spaces, Ph.D. thesis, Meerut University, Institute of advanced studies, Meerut-India, (1980).
4. N. Levine, Semi-open sets and semi-continuity in topological space , Amer. Math. Monthly., *70*, 39-41 (1963).
5. S. Malathi and S. Nithyanantha Jothi, On c*-open sets and generalized c*-closed sets in topological spaces, Acta ciencia indica, Vol. XLIII M, No.2, 125, 125-133 (2017).
6. S. Malathi and S. Nithyanantha Jothi, On generalized c*-open sets and generalized c*-open maps in topological spaces, Int. J. Mathematics And its Applications, Vol. *5,* issue 4-B, 121-127 (2017).
7. S. Malathi and S. Nithyanantha Jothi, On generalized c*-continuous functions and generalized c*-irresolute functions in topological spaces, Turkish Journal of Analysis and Number Theory, (Accepted).
8. S. Malathi and S. Nithyanantha Jothi, On pre-generalized c*-closed sets in topological spaces, *Journal of Computer and Mathematical Sciences* Vol. *8 (12),* 720-726 (2017).
9. S. Malathi and S. Nithyanantha Jothi, On pre-generalized c*-open sets and pre-generalized c*-open maps in topological spaces, Int. J. Mathematical Archive, Vol. *8 (12)*, 66-70 (2017).
10. S. Malathi and S. Nithyanantha Jothi, On pre-generalized c*-continuous functions and pre-generalized c*-

irresolute functions in topological spaces, Mathematical Sciences *International Research Journal*, Vol. *7,* Spl issue 4, 17-22 (2018).

11. S. Malathi and S. Nithyanantha Jothi, On generalized c*-neighbourhoods and generalized c*-homeomorphisms in topological spaces, Proceedings of National Conference on Innovation in Mathematics (NCIM-2018), 87-95.
12. A.S. Mashhour, M.E. Monsef and S.N. El-Deep, On precontinuous mapping and weak precontinuous mapping, Proc. Math. Phy. Soc. Egypt, *53,* 47-53 (1982).
13. M. Stone, Application of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., *41,* 374-481 (1937).
14. P. Sundaram and M. Sheik John, Weakly closed sets and weak continuous functions in topological spaces, Proc. 82nd Indian Sci. cong., *49*, 50-58 (1995).
15. P. Sundaram, M. Sheik John, On w-closed sets in topology, Acta ciencia indica, *4,* 389-392 (2000).
16. M.K.R.S. Veera kumar, On $\hat{g}$ –closed sets in topological spaces, Bull. Allah. Math. Soc, *18,* 99-112 (2003).