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Network Flow in Graph Theory - The Producer's Problem

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Abstract

The maximum flow problem was first formulated in 1954 by T.E. Harris and F.S. Ross as a simplified model of soviet Railway traffic flow[1]. In 1955, Lester R. Ford, Jr. and Delbert R. Fulkerson created the first known algorithm for calculating the maximum flow, the Ford-Fulkerson algorithm[2]. It was in 1951 when the American mathematician, George Dantzig put forward the network simplex algorithm to solve mininimum cost flow problem. We take a typical example and find the maximum flow and minimum cost. Finally we generalise ***The Producer's Problem*** based on some assumptions and find the maximum flow and minimum cost in it. We observe that the minimum unit cost in ***The Producer's Problem*** increases with the number of items produced increases, in contrast to natural expectation.

Key words: Network, flow, feasible flow, flow value, s-t cut, augmenting path, leeway, maximum flow.

Subject classification: 05C21

1 Introduction

Network Theory is one of the most dynamic and exciting areas of science today. The language of networks is graph theory. Translating a given problem or a situation comprising of large data, into a network makes the problem more comprehensive and easy to handle. Thus learning to model and design networks is essential for the progress of science and engineering, in the 21st century.

In this paper, the relevent facets of a network like maximum flow and minimum cost are discussed and finally, an attempt is made to address, formulate and solve ***'The Producer's Problem'***.

2 Basic Concepts :

A ***network*** N(s,t) is a digraph with two distinguished vertices, a source s and a sink t, together with a non-negative function c(a) defined on the arc set of the digraph is called a network.

A ***Flow*** f in a network N(V,A) is a real valued function defined on the set of arcs of the network satisfying the *conservation condition* which states that the flow into a vertex is equal to the flow out of a vertex, for all intermediate vertices; ie,

$$f^{+}(v) = f^{-}(v), \forall v \in V - \{s,t\}$$

When the flow function satisfies the *capacity condition*, which requires that the value of the flow on every edge is bounded by the capacity on that edge, it is called a *feasible flow* of the network. Thus a feasible flow must satisfy,

$$0 \leq f(a) \leq c(a), \forall a \in A$$

Flow value is the rate at which we can send a commodity from source to the sink ie, the amount of flow passing from the source to the sink. It is denoted by *val (f)*. An ***s-t cut*** of network N is a partition of the vertices V into two sets S and $\overline{S} = V - S$ such that the source and sink of the network are in separate sets, ie,

$s \in S$ and $t \in \overline{S}$.

It is denoted by $[S, \overline{S}]$ or simply by S. The *net flow* across a cut $[S, \overline{S}]$ or S is defined as the sum of the flows on the edges in the cut set, ie,

$$f(S) = \sum_{v \in S} \sum_{w \in \overline{S}} f(v,w)$$

The ***value or capacity*** of a cut is defined as sum of the capacities of the edges in the cut set that has initial vertex in the set containing the source and terminal vertex in the set containing the sink, ie,

$$u(S) = \sum_{v \in S} \sum_{w \in \overline{S}} c(vw)$$

Let f be a feasible flow on a network N. The corresponding ***residual network***, denoted N_f, is a network that has the same vertices as the network N, but has edges with capacities $c_f(vw) = c(vw) - f(v,w)$. Only edges with non-zero capacity, $c_f(vw) \gneq 0$, are included in N_f. An ***augmenting path*** is a directed path from the node s to node t in the residual network N_f. It may contain both forward and backward edges. A path in a network becomes an augmenting path when all forward edges in the path have excess capacity and all the backward edges have non-zero flow. Note that if we have an augmenting path in N_f, then this means we can push more flow along such a path in the original network N.

The ***leeway*** of the path P in a network is the minimum of the excess capacity $c(e) - f(e)$ on all forward edges of P and the non-zero flow $f(e)$ on all backward edges of P. By the definition of f-augmenting path, the leeway of such a path is strictly positive.

2.1 Lemma[3]:

If U is a set of nodes in a network, then the net flow out of U is the sum of the net flow out of the nodes in U. In particular, if f is a feasible flow $[S, \overline{S}]$ is a source sink cut, then the net flow out of S and the net flow into $\overline{S}$ equals *val f*.

2.2 Corollary[3]*:*

If f is a feasible flow and $[S,\overline{S}]$ is a source sink cut, then:

$$val\, f \leq u(S,\overline{S}).$$

This is an important observation because, if we are in search of maximum flow we need to search for the flow which has value equal to capacity of some cut,which will be the minimum cut. We can thus identify the maximum flow by checking for a saturated cut, ie a cut through which flow cannot be incremented.

2.3 Lemma[3]*:*

If P is an f-augmenting path with leeway ε then increasing flow by ε along the forward edges of P and decreasing flow by ε along the backward edges of P produces a feasible flow f^* with value $val(f^*) = val(f) + \varepsilon$.

3 Maximum Flow :

The idea of maximum flow is worth pursuing, because any network is built with the aim of transporting commodities by introducing a flow function. If the flow function is such that it reduces the time and cost in carrying the initial flow at the source to the sink, it will be desirable. Hence maximising the flow function without violating the conservation and capacity constraint, ie, maintaining its status as a flow, will increase the efficiency of transportion. Infinite flow can only be carried through edges of infinite capacity, which is practically impossible to build. An algorithm designed by L.R. Ford and D.R. Fulkerson Jr. to find the maximum flow in a network which is described below.

3.1 Ford-Fulkerson Algorithm[3,6]*:*

- **Input**: A feasible flow f in the network.
- **Output**: An f-augmenting path or a cut with with capacity $val(f)$.
- **Idea**: Finding the nodes reachable from the source s by paths having positive leeway reaching t completes an f augmenting path. During the search,we collect nodes into R and S; where R is the set of nodes we have reached and S is the subset of these from which we have searched to extend R.
- **Initialisation**: $R = s, S = \phi$
- **Iteration**: Choose $v \in R - S$. Consider each edge vw leaving v and each edge uv entering v. If vw has excess capacity $(f(vw) \geq c(vw)$, include w in R. If uv has nonzero flow $(f(uv) \geq 0)$, include in R. Record v as the vertex from which the new vertex of R was reached. After exploring all edges involeving v, add v to S. If the sink t has been put in R, trace back the path of inclusions leading to $t \in R$ to return an f-augmenting path. If $R = S$, terminate and return the cut $[S,\overline{S}]$. Otherwise iterate.

3.2 Theorem[3,4]*:*

In every network, the maximum value of feasible flow equals the minimum capacity of a source sink cut; ie; in a flow network G, the following conditions are equivalent:

1. A flow f is a maximum flow.
2. The residual network Gf has no augmenting paths.
3. $val(f) = u(S)$ where $u(S)$ is the capacity of some cut S.

These conditions imply that the value of the maximum flow is equal to the value of the minimum $s-t$ cut.

This theorem helps us to find the maximum flow in a network by searching for the cut with minimum capacity.

4 Minimum Cost Flow :

In a network if we assign a cost to each of the arcs, we would have a problem to find the minimum cost of transporting an item from the source to the sink. We may develop the linear programming problem corresponding to this with the objective of minimising the cost and then solve. In the following sections we discuss an algorithm to find the minimum cost.

4.1 Algorithm[5] *:*

- Set: $x_{ij} = 0$, $e_{ij} = u_{ij}; \forall arcs(v_i, v_j)$ and $G = 0$, where G denotes the $v_1 - v_n$ flow assigned so far.
- Identify the $v_1 - v_n$ path, of least cost with positive excess capacity,E.

  For each arc (v_i, v_j) on this path set,

  E to become min $\{E, b - G\}$

  e_{ij} to become $e_{ij} - E$

  x_{ij} to become $x_{ij} + E$

  c_{ij} to become ∞ if $e_{ij} = 0$

  G to become $G + E$

  For each arc (v_j, v_i) oppositely directed to arc (v_i, v_j) just identified set e_{ji} to become $e_{ji} + E$ and c_{ji} to become $-c_{ij}$. If for any pair of arcs (v_i, v_j) and (v_j, v_i) it happens that $x_{ij} \geq 0$ and $x_{ji} \geq 0$ set: x_{ij} to become $x_{ij} - | x_{ij} - x_{ji} |$

  x_{ji} to become $x_{ji} - | x_{ij} - x_{ji} |$

  e_{ij} to become $e_{ij} + | x_{ij} - x_{ji} |$

  e_{ji} to become $e_{ji} + | x_{ij} - x_{ji} |$
- If $G \leq b$, go to step (ii). If $G = b$, terminate as the current x_{ij} values indicate a least cost solution.

5 Dependence of maximum flow and minimum cost in a network :

In a particular network, we change flow functions and try to find the maximum flow and check whether the cost corresponding to this flow is the minimum or not.

5.1 Problem 1 :

Suppose that a manufacturer intends to produce some quantity of an item. In the production of that item many stages are invovled. Suppose that there are many factories that render the same service at every stage of production, and each factory has an upper bound for the number of items it can process and also a fixed cost per unit is involved for processing at each factory. For each unit being processed in a factory, a production charge is incurred to the manufacturer. In the following network, we find the maximum flow possible in the network and also check for the minimum cost flow.

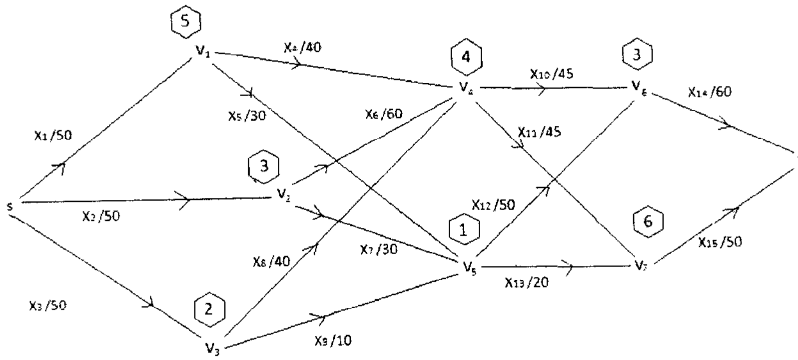


Figure 1: Network of the problem.

Using Ford-Fulkerson algorithm we find the maximum flow to be 110. The unit cost incurred is 9.54 rupees. Changing the flow functions and finding minimum cost for each we reach the conclusion that as the flow increases the minimum unit cost also increases.

5.2 Producer's Problem :

Every producer or manufacturer in the economy possesses a limited amount of resources required for production ie, raw materials, service and capital. Let us consider the following problem faced by a producer who has to work his way through the raw materials at hand, service available and his investment to maximise his profit by minimising the cost involved. His cost precisely involves freight charge as well as unit cost of production at each factory. As seen in the above problem, at every stage of manufacturing, there are several production units or factories that carry out the same process but which has different cost of production and freight charges per unit. The producer in this problem desires to select a rational strategy in order to minimise his total cost of producing a fixed number of items in a given set-up. We try to solve this problem using linear programming.

Translating this problem into a graph theoretic model, we may take the producer to be at the vertex x $(source)$ where the raw materials required for the production of a fixed number, C is available. Let $S_1, S_2, ..., S_l$ denote the stages of production. Let $U_{11}, U_{12}, ..., U_{1n_1}$ denote the different factories available at stage 1; $U_{21}, U_{22}, ..., U_{2n_2}$ denote those of S_2,... until $U_{l1}, U_{l2}, ..., U_{ln_l}$ denote those of S_l. Each production unit has an upper limit on the number items it can produce which is its capacity of production. Also the unit cost of production for each factory is different, which is split into production cost and freight cost. Let (k_{ij}, c_{ij}) denote the capacity and the unit production cost respectively of a factory U_{ij} where $i = 1,2,3...,l$ and $j = n_1, n_2, ..., n_l$. Let t_{ij} denote the freight charge per unit.

5.2.1 Assumptions:

The assumptions involved in this problem are: 1. The manufacturing process must produce the stipulated number of items that was aimed at the beginning of production, assuming that no loss or defects occur during production.

2. At every factory the number of items demanded by the producer must be within the limit of the number of items that the factory can produce. This means that the producer must keep in mind, no factory can process greater number of items than its capacity.
3. The number of items demanded from each factory must be met by it, once the producer has placed the order. In short the factory is obliged to process the number of items demanded of it.
4. In this problem, the cost is associated with the process of production alone, which means that the cost of the raw materials at hand is met and hence is ignored.

5.2.2 Integer Linear Programming Formulation :

Objective function is to minimise

$$z = \sum_j x_{1j}(t_{1j} + c_{1j}) + \sum_{r,s}\sum_{p,q} x_{pq}(r,s)(c_{pq} + t_{pq}(r,s)) + \sum_k x_{lk}t_{lk}$$

Such that:

$$\sum_j x_{1j} = C;$$

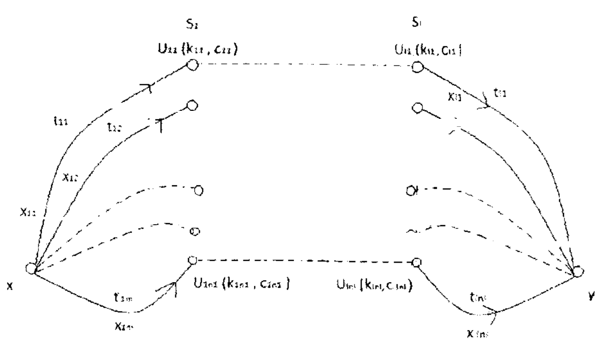
$$\sum_k x_{lk} = C;$$

$$\sum x_{pq}(r,s) \le k_{pq}; \text{ where } p = r+1, p = 2,3,...,l \text{ and } q = 1,2,...,n_p$$

$$x_{1j} < k_{1j}; \forall j;$$

$$\sum_{s,r=p-1} x_{pq}(r,s) = \sum_{s,r=p+1} x_{rs}(p,q)$$

5.2.3 Graphical Representation :



Here the variables are defined as:

- C = fixed number of items aimed to produce.
- U_{ij} = j *th* production unit at the i *th* stage $(i = 1,2,...,l \;;\; j = n_1, n_2, ..., n_l)$
- t_{1j} = freight charge from x to U_{1j}; $j = 1,2,..., l$
- $t_{pq}(r,s)$ = freight charge between U_{rs} to U_{pq} where $r = 1,2,3,..., (l-1); s = 1,2,..., n_r;$ $p = 2,3,..., l; q = 1,2,..., n_p andp = r+1$
- t_{lk} = freight charge from U_{lk} to destination y, $k = 1,2..., n_l$.

- $S_i =$ i^{th} stage $i = 1,2,...,l$
- $x_{1j} =$ number of items to be produced by U_{1j} where $j = 1,2,..., n_1$
- $x_{pq}(rs) =$ number of items sent from U_{rs} to U_{pq}
- $x_{lk} =$ finished goods from U_{lk} to destination y, where $k = 1,2,...,n_l$
- $k_{ij} =$ capacity of U_{ij}
- $c_{ij} =$ unit cost of production at U_{ij}

6 Example:

Consider a particular example of the above problem. (The notations are defined as per the definitions in the general case).

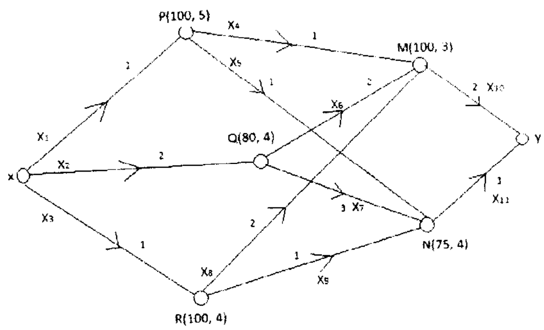


Figure 2: Example.

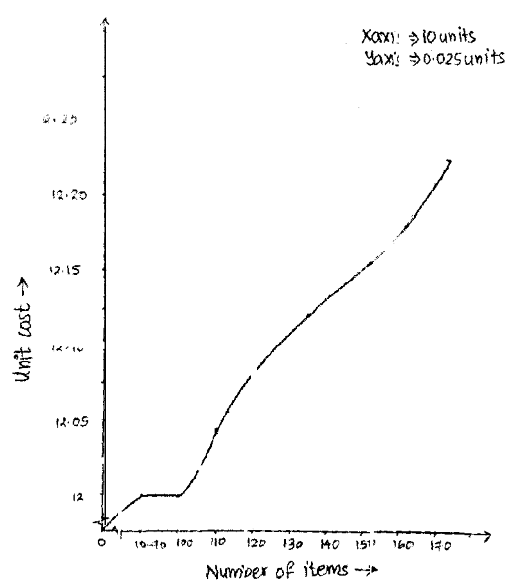


Figure 3: Unit cost graph for different number of items.

Changing the number of items to be produced and finding the minimum unit cost, we plot a graph. We find that the unit cost is proportional to the number of items produced, ie; as the number of items to be produced in a network increases the unit cost of production also increases. We can also find the optimal assignment of the order to be placed at each factory, by formulating the ILPP from the generalised equations given above and solving it using LP Solve software. An aspiring producer can use this information to calculate the costs involved in producing a certain number of items in a given network and can choose the strategy that suits his budget.

7 Conclusion

In this paper, we have gone through some of the basic theorems and results that help in modelling and solving real life problems using networks. It has been explicitly shown how to translate a cost minimisation problem to networks and solve it. It has also elaborated The Producer's Problem, which deals with the concerns of a producer of how can he maximise the production at minimum cost. The problem aims at finding an equilibrium in the budget and aspiration of the producer. A generalised mathematical formulation of the problem has been given and the solution has been discussed. By the use of this generalisation, a producer can foresee the effects of investing effort and money to different strategies available in a given production scenario and can judiciously plan the manufacturing process according to his budget. This saves the manufacturer from falling into potential loss or risk and he can also make the most of the available resources. At the end, it has been observed that, unlike the real-life production processes, where the unit cost of production decreases as the production increases, The Producer's Problem that we have formulated shows direct proportionality between unit cost and production.

8 Scope of further research :

In the 21^{st} century, where networks rule every complex and intriguing processes, the study and comprehension of their logic is very important. It has been an area of meticulous research and progress. The more we are equipped with insight in the behaviour of networks, the more efficiently can we handle complex problems.

This work can be extended to include new assumptions and constraints. This concept can be elaborated to calibrate the stability (by calculating income and expenditure) of an economy.

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