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Section A

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Multivariable H-Function and Problem Related to *Flux* Condition

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Abstract

The aim of this research paper is to derive the solution of a problem related to *flux* condition involving the multivariable H-function.

Key words: H-Function, Flux condition.

1. Introduction

An insulation-type flux condition at, say, $x = 0$ is a condition on the derivative of the form $u_x(0, t) = 0$, $t > 0$; because the flux is proportional to the temperature gradient (Fourier's heat flow law states flux = $-Ku_x$, where K is the conductivity), the insulation condition requires that no heat flow across the boundary $x = 0$. A radiation condition, on the other hand, is a specification of how heat radiates from the end of the bar, say at $x = 0$, into its environment, or how the end absorbs heat from its environment. Linear, homogeneous radiation conditions take the form $-Ku_x(0, t) + bu(0, t) = 0$, $t > 0$, where b is a constant. If $b > 0$, then the heat flux is negative, which means that heat is flowing from the bar into its surroundings (radiation); if $b < 0$, then the flux is positive, and heat is flowing into the bar (absorption). A typical problem in heat conduction may have a combination of Dirichlet, insulation, and radiation boundary conditions. In other physical contexts, e.g., contaminant transport, biological diffusion, these three types of boundary conditions also play an important role.

The multivariable H–function given in[4] is defined as follows:

$$H[z_1, \dots, z_r] = H^{0,n:m_1,n_1;\dots;m_r,n_r}_{p,q:p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix}(a_j;\alpha'_j,\dots,\alpha^{(r)}_j)_{1,p}:(c'_j;\gamma'_j)_{1,p_1};\dots\dots;(c^{(r)}_j;\gamma^{(r)}_j)_{1,p_r} \\ (b_j;\beta'_j,\dots,\beta^{(r)}_j)_{1,q}:(d'_j;\delta'_j)_{1,q_1};\dots\dots;(d^{(r)}_j;\delta^{(r)}_j)_{1,q_r}\end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\dots\int_{L_r}\phi_1(\xi_1)\dots\phi_r(\xi_r)\psi(\xi_1,\dots,\xi_r)z_1^{\xi_1}\dots z_r^{\xi_r}d\xi_1\dots d\xi_r \tag{1}$$

where $\omega = \sqrt{(-1)}$,

$$\psi(\xi_1,\dots,\xi_r) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)}{\prod_{j=n+1}^{p}\Gamma(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)\prod_{j=1}^{q}\Gamma(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i)}$$

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\xi_i)}$$

In (1), i in the superscript (i) stands for the number of primes, e.g., $b^{(1)} = b'$, $b^{(2)} = b''$, and so on; and an empty product is interpreted as unity.

Suppose, as usual, that the parameters

a_j, j = 1, …., p; $c_j^{(i)}$, j = 1,….,p_i;

b_j, j = 1, …., q; $d_j^{(i)}$, j = 1,….,q_i; $\forall i \in \{1,…..,r\}$

are complex numbers and the associated coefficients

$\alpha_j^{(i)}$, j = 1, …., p; $\gamma_j^{(i)}$, j = 1,….,p_i;

$\beta_j^{(i)}$, j = 1, …., q; $\delta_j^{(i)}$, j = 1,….,q_i ; $\forall i \in \{1,…..,r\}$

positive real numbers such that the left of the contour. Also

$$V_i = \sum_{j=1}^{p} \alpha_j^{(i)} + \sum_{j=1}^{p_i} \gamma_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} - \sum_{j=1}^{q_i} \delta_j^{(i)} \leq 0 \tag{2}$$

$$\Omega_i = -\sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} > 0 \tag{3}$$

where the integral n, p, q, m_i, n_i, p_i and q_i are constrained by the inequalities $p \geq n \geq 0$, $q \geq 0$, $q_i \geq m_i \geq 1$ and $p_i \geq n_i \geq 1$ □ $i \in \{1, 2, …, r)$ and the inequalities in (2) hold for suitably restricted values of the complex variables z_1, …., z_r. The sequence of parameters in (1) are such that none of the poles of the integrand coincide, that is, the poles of the integrand in (1) are simple. The contour L_i in the complex ξ_i-plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $+\omega\infty$ with indentations, if necessary, to ensure that all the poles of $\Gamma\,(d_j^{(i)} - \delta_j^{(i)}\xi_i)$, j = 1,…, m_i are separated from those of $\Gamma\,(1 - c_j^{(i)} + \gamma_j^{(i)}\xi_i)$, i = 1, …., n_i.

In the present investigation we require the following formula:

From Gradshteyn [1, p.372]:

$$\int_0^1 (\sin\frac{\pi x}{l})^{\omega-1} \cos\frac{(n+\frac{1}{2})\pi x}{l}\, dx$$

$$= \frac{l \cos\left(\frac{n}{2}+\frac{1}{4}\right)\pi\Gamma(\omega)}{2^{\omega-1}\Gamma[\frac{1}{2}\{\omega \pm \left(n+\frac{1}{2}\right)+1\}]}, \tag{4}$$

where Re (ω) > 0.

From Rainvile[3], Legendre's duplication formula:

$$\sqrt{\pi}\, \Gamma\,(2z) = 2^{2Z-1}\Gamma(z)\, \Gamma(z + ½). \tag{5}$$

2. Integral :

The following integral is required in this paper:

$$\int_0^l (sin\frac{\pi x}{l})^{\omega-1} cos\frac{(n+\frac{1}{2})\pi x}{l}$$

$$\times H^{0,\ n\ :(m_1,n_1);\ldots\ldots\ldots;\ (m_r,n_r)}_{p,\ q\ :(p_1,q_1)\ ;\ldots\ldots\ldots;\ (p_r,q_r)}\left[\begin{matrix} z_1(sin\frac{\pi x}{l})^{-2\rho} \\ \vdots \\ z_r \end{matrix}\right]dx$$

$$= \frac{l}{\sqrt{\pi}}.H^{0,\ n\ :(m_1+2,n_1);\ldots\ldots\ldots;\ (m_r,n_r)}_{p,\ q\ :(p_1+2,q_1+2)\ ;\ldots\ldots\ldots;\ (p_r,q_r)}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\right|$$

$$\left.\begin{matrix} \ldots\ldots,\ldots:\ldots\ldots,(\frac{1}{2}+\frac{\omega}{2}\pm\left(\frac{n}{2}+\frac{1}{4}\right),\rho).:\ldots\ldots,\ldots\ldots \\ \ldots\ldots,\ldots:\left(\frac{\omega}{2},\rho\right),(\frac{1}{2}+\frac{\omega}{2},\rho)\ldots\ldots:\ldots\ldots,\ldots\ldots \end{matrix}\right], \tag{6}$$

provided that ρ > 0, $|\arg(z_k)| < \frac{1}{2}V_k\pi, \forall k \in [1, \ldots, r]$, where V_k is given in (2).

The integral (6) can be obtained easily in routine manner by making use of the definition of multivariable H-function as given in (1), (4) and (5).

3. Problem Related To Flux Condition:

In this section, we consider a problem on flux condition under certain boundary conditions.

Consider the diffusion problem

$$u_t = ku_{xx},\ 0 < x < l,\ t > 0, \tag{7}$$

$$u_x(0, t) = 0,\ u(l, t) = 0,\ t > 0, \tag{8}$$

$$u(x, 0) = f(x),\ 0 < x < l, \tag{9}$$

where the left end is insulated and the right end is fixed. The solution of (7) – (9) is given by [2, p. 129-131] as follows:

$$u(x, t) = \sum_{n=0}^{\infty} c_n e^{-\lambda_n kt} \cos\frac{(2n+1)\pi x}{2l}, \tag{10}$$

where

$$c_n = \frac{2}{l}\int_0^l f(x)\cos\frac{(2n+1)\pi x}{2l}\,dx,\ n = 0, 1, 2, \ldots. \tag{11}$$

4. Solution Of The Problem:

The solution of the problem to be obtained is

$$u(x, t) = \sum_{n=0}^{\infty} e^{-\lambda_n kt}\cos\frac{(2n+1)\pi x}{2l}$$

$$= \frac{2}{\sqrt{\pi}}.H^{0,\ n\ :(m_1+2,n_1);\ldots\ldots\ldots;(m_r,n_r)}_{p,\ q\ :(p_1+2,q_1+2);\ldots\ldots\ldots;(p_r,q_r)}\left[\begin{matrix} z_1 \\ \vdots \\ z_r\end{matrix}\right| \begin{matrix}\ldots\ldots,\ldots:\ldots\ldots,\left(\frac{1}{2}+\frac{\omega}{2}\pm\left(\frac{n}{2}+\frac{1}{4}\right),\rho\right).:\ldots\ldots,\ldots\ldots \\ \ldots\ldots,\ldots:\left(\frac{\omega}{2},\rho\right),\left(\frac{1}{2}+\frac{\omega}{2},\rho\right)\ldots:\ldots\ldots,\ldots\ldots\end{matrix}\right], \tag{12}$$

where all conditions of convergence are same as in (6).

Proof: Choose

$$f(x) = \left(\sin\frac{\pi x}{l}\right)^{\omega-1} H^{0,\ n\ :(m_1,n_1);\ldots\ldots\ldots;(m_r,n_r)}_{p,\ q\ :(p_1,q_1);\ldots\ldots\ldots;(p_r,q_r)}\left[\begin{matrix} z_1\left(\sin\frac{\pi x}{l}\right)^{-2\rho} \\ \vdots \\ z_r\end{matrix}\right] \tag{13}$$

Now combining (13) and (11) and making the use of the integral (6), we derive

$$C_n = \frac{2}{\sqrt{\pi}}.H^{0,\ n\ :(m_1+2,n_1);\ldots\ldots\ldots;(m_r,n_r)}_{p,\ q\ :(p_1+2,q_1+2);\ldots\ldots\ldots;(p_r,q_r)}\left[\begin{matrix} z_1 \\ \vdots \\ z_r\end{matrix}\right| \begin{matrix}\ldots\ldots,\ldots:\ldots\ldots,\left(\frac{1}{2}+\frac{\omega}{2}\pm\left(\frac{n}{2}+\frac{1}{4}\right),\rho\right).:\ldots\ldots,\ldots\ldots \\ \ldots\ldots,\ldots:\left(\frac{\omega}{2},\rho\right),\left(\frac{1}{2}+\frac{\omega}{2},\rho\right)\ldots:\ldots\ldots,\ldots\ldots\end{matrix}\right], \tag{14}$$

Putting the value of C_n from (14) in (10), we get the required result (12).

5. Special Cases:

On specializing the parameters, multivariable H-function may be reduced to H-function, G-function, Lauricella's functions Legendre functions, Bessel functions, hypergeometric functions, Appell's functions, Kampe de Feriet's functions and several other higher transcendental functions. Therefore the result (14) is of general nature and may reduced to be in different forms, which will be useful in the literature on applied Mathematics and other branches.

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