ISSN 2231-346X (Print)

JUSPS-A Vol. 32(3), 13-21 (2020). Periodicity-Monthly

(Online) ISSN 2319-8044

Section A

JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES

An International Open Free Access Peer Reviewed Research Journal of Mathematics

website:- www.ultrascientist.org

Estd. 1989

Existence of Smooth Epimorphism from Fuchsian Group to the Group of Automorphisms of compact Riemann surface to the point group of Carbon Tetrachloride

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AMS Subject Classification 2020: 20B25, 20B30, 20H10, 30F10

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http://dx.doi.org/10.22147/jusps-A/320301

Acceptance Date 07th June, 2020, Online Publication Date 09th June, 2020

Abstract

A finite group G acts as a group of automorphisms on a compact Riemann surface S of genus g if and only if there exist a Fuchsian group Γ and an epimorphism $\phi:\Gamma\rightarrow G$ such that $ker\phi = K$ is a surface group of genus g. And then ϕ is named as smooth homomorphism. The objective of this paper is to establish a set of necessary and sufficient conditions for the existence of smooth epimorphism from a Fuchsian group Γ to the finite group of symmetries of Carbon Tetra chloride molecule, whose abstract group representation is $\langle a,b|a^4= b^3=(ab)^2\rangle$.

Key words : smooth epimorphism, point group, compact Riemann surface, group of automorphisms.

Introduction

Historically the study of birational transformations of algebraic curves began in the last quarter of the nineteenth century, and this topic attracted the attention of a number of renowned mathematicians of the time, like H.A. Schwarz, A. Hurwitz, A.Wiman, W. Burnside etc. [3]. The famous result of Burnside- "Every finite group can be realized as a group of transformations of an algebraic curve of genus $g \geq 2$." had given a new dimension in this topic,which had lead to the proposition that an algebraic curve could be conceived of as a compact Riemann surface together with its birational transformations, so that this combination can be realized as biholomorphic self transformations, usually called autmorphism of such surface.[1,2,7,5,6,9] In this manner the study of automorphisms of algebraic curve had been redirected to the study of compact Riemann surfaces. The first significant results on the topic were published by H.A. Schwarz in 1879[12]. Further study in the topic in the new garb as automorphisms of compact Riemann surfaces was initiated afresh after a long gap in the early sixties of last century by A.M. Macbeath and his students. In his famous Dundee Summer School Lectures, A.M. Macbeath formulated a number of interesting broad problems on this topic and announced partial solutions to some of them[18].

The theory of Fuchsian group is intimately related to the theory of the groups of automorphims of compact Riemann surfaces [10,11,12,16,17]. A Fuchsian group is an infinite group generated by k elements $x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots x_k$ of finite orders $m_1, m_2, \ldots\ldots\ldots\ldots\ldots\ldots, m_k$ respectively and 2γ elements $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \alpha_\gamma, \beta_\gamma$, of infinite order with defining relations:

$$\langle x_1, x_2, \ldots\ldots, x_k\,;\, b_1, c_1, b_2, c_2, \ldots\ldots, b_\gamma, c_\gamma;\, x_1^{m_1} = x_2^{m_2} = \prod_{i=1}^{k} x_i \prod_{j=1}^{\gamma}\left[b_j, c_j\right] = 1 \rangle \ldots.. \qquad [1]$$

Where $[b_j c_j] = b_j^{-1} c_j^{-1} b_j c_j$ and $\delta(\Gamma) = 2\gamma - 2 + \sum(1 - \frac{1}{m_i}) > 0$ [2]

The group of symmetries of the molecular structure of any non-linear molecule is a very beautiful finite group which can be considered as the particular finite group to which a smooth epimorphism exist from a Fuchsian group Γ[4,5,13,14]. Here we consider the molecule, carbon tetrachloride whose symmetry structure possesses the point group-

$$T_d = \{E, 4C_3, 4C_3^{'}, 3C_2, 3S_4, 3S_4', 6\sigma_d\}.$$

And the abstract presentation of this point group is-

$$G(\ CCl_4), = \langle a, b | a^4 = b^3 = (ab)^2 \rangle \qquad [3]$$

Group representation:

Table 1. Group presentation of G(CCl_4),

Sl no	Symmetry elements	permu tation	Alpha symb	presentation	Sl no	Symmetry elements	permu tation	Alpha symb	prese-ntation
1	E	1234	E	$a^4 = b^3 = (ab)^2$	13	S_4^1	2314	N	a^2b^2
2	C_3^1	1423	A	b^2	14	S_4^2	4312	0	a^3b^2a
3	C_3^2	2413	B	ba^3	15	S_4^3	3421	P	a^3b
4	C_3^3	3124	C	a^3b^2a	16	S'^1_4	2431	Q	ab^2a
5	C_3^4	4132	D	ba^2b	17	S'^2_4	4123	R	a
6	C'^1_3	1342	F	b	18	S'^3_4	3142	S	ab^2
7	C'^2_3	2341	G	a^3	19	σ_d^1	1243	T	a^2b^2a
8	C'^3_3	3241	H	b^2a^2	20	σ_d^2	4231	U	ba

9	C'^4_3	4213	J	ab	21	σ^3_d	1324	V	ab^2a^2
10	C^1_2	2143	K	ba^2b^2	22	σ^4_d	1432	W	ba^2b^2a
11	C^2_2	4321	L	b^2a^2b	23	σ^5_d	2134	X	ab
12	C^3_2	3412	M	a^2	24	σ^6_d	3214	Y	ba^3b
Presentation: $\langle a, b \mid a^4 = b^3 = (ab)^2 \rangle$									

Theorem: The group of symmetries G(CCl_4), of the Carbon tetra-chloride molecule CCl_4 can be acted as a group of Automorphism of some Riemann Surface S of genus $g(\geq 2)$ then there is a smooth epimorphism $\phi : \Gamma \to$ G(CCl_4), if and only if the following conditions are satisfied-

1. When $k = 0$ i.e. $\Gamma = \Delta(\gamma : -)$ a surface group then ≥ 2.
2. When $k \neq 0$, $\phi(x_i) = m_i$, m_i divides 24.

Moreover

(i) If all $\phi(x_i) \in \langle a \rangle$ then m_i divides 4i.e. $m_i = 2,4$ And $\gamma \geq 1$.

(ii) If all $\phi(x_i) \in \langle b \rangle$ then m_i divides 3 and $\gamma \geq 1$.

(iii) If all $\phi(x_i) \in \langle ab \rangle$ then m_i=2, k is even and $\gamma \geq 1$.

(iv) If some $\phi(x_i) \in \langle a \rangle$ for $i = 1,2, \ldots, s; s < k$ and remaining $(x_{s+j}) \in \langle b \rangle$, $j = 1,2, \ldots, t$. Then $s + t = k$, s must be even so $k \geq 3$

(v) If some $\phi(x_i) \in \langle a \rangle$ for $i = 1,2, \ldots, s; s < k$ and remaining $\phi(x_{s+j}) \in \langle ab \rangle$, $j = 1,2, \ldots, t$. Then $s + t = k$, s, t even and so $k \geq 4$

(vi) If some $\phi(x_i) \in \langle b \rangle$ for $i = 1,2, \ldots, s; s < k$ and remaining $\phi(x_{s+j}) \in \langle ab \rangle$, $j = 1,2, \ldots, t$. Then $s + t = k, t$ is even and so $k \geq 3$

(vii) If some $\phi(x_i) \in \langle a \rangle$ for $i = 1,2, \ldots, s; s < k$, some $\phi(x_{s+j}) \in \langle b \rangle, j = 1,2, \ldots, t$, and remaining $(x_{s+t+j}) \in \langle ab \rangle$, $j = 1,2, \ldots, p$ such that $s + t + p = k, p$ is even, $t + p$ is even if t is even and $t + p$ is odd if t is odd.

(a) If $t + p$ is even then so $k \geq 6$
(b) If $t + p$ is odd then so $k \geq 5$

Proof: Let $\phi : \Gamma \to$ G(CCl_4) be a smooth epimorphism, then we see that –

(1) If $k = 0$ then from[1], the measure of Fuchsian group it is clear that $2\gamma - 2 > 0$
I.e. if $k = 0$ then $\gamma \geq 2$.

Therefore the condition is necessary.

Again the condition is sufficient because, if $k = 0$, $\gamma \geq 2$ then we can define $\phi : \Gamma \to$ G by

$$\phi(\alpha_1) = a = \phi(\alpha_2)$$
$$\phi(\alpha_3) = a^2 = \phi(\alpha_4)$$
$$\phi(\alpha_5) = a^3 = \phi(\alpha_6)$$
$$\phi(\beta_1) = \phi(\beta_3) = \phi(\beta_5) = b$$

$\phi(\beta_2) = \phi(\beta_4) = \phi(\beta_6) = b^2$

$\phi(a_j) = 1 \ \phi(\beta_j); 4 \le j \le \gamma$

Then $\prod_{j=1}^{\gamma}[\phi(\alpha_j), \phi(\beta_j)]=[\alpha_1,\beta_1].[\alpha_2,\beta_2].[\alpha_3,\beta_3].[\alpha_4,\beta_4].[\alpha_5,\beta_5].[\alpha_6,\beta_6].$

$= [a, b]. [a, b^2]. [a^2, b]. [a^2, b^2]. [a^3, b]. [a^3, b^3]$

$= (a^{-1}. b^{-1}.a.b). (a^{-1}. b^{-2}.a.b^2).(a^{-2}. b^{-1}.a^2.b).$

$(a^{-2}. b^{-2}.a^2.b^2). (a^{-3}. b^{-1}.a^3.b). (a^{-3}. b^{-2}.a^3.b^2)$

$= a^{-2}. b^2.a^{-2}.b^4. a^{-4}. b^2.a^{-4}.b^4.a^{-6}. b^2.a^{-6}.b^4$

$= a^{-2}. a^2.b^{-2}.b^4. a^{-4}. a^4.b^{-2}.b^4.a^{-6}. a^2.b^{-2}.b^4$

$= 1. b^2. 1.b^2. 1.b^2$

$= b^3. b^3$

$= 1$

Then clearly ϕ is a homomorphism as $\prod_{j=1}^{\gamma}[\phi(\alpha_j), \phi(\beta_j)] = 1$

Also $a, b \in \phi(\Gamma)$ gives ϕ is a smooth epimorphism.

(2) When $k \neq 0$, $\phi(x_i) = m_i$, m_i divides 24.

(i) If all $\phi(x_i) \in \langle a \rangle$ then clearly m_i divides 4. If $\gamma = 0$ then $\phi(\Gamma) = \langle a \rangle$ which gives ϕ is not onto which contradicts our assumption that ϕ is onto, hence if all $\phi(x_i) \in \langle a \rangle$ then $\gamma \ge 1$.

(ii) If all $\phi(x_i) \in \langle b \rangle$ then clearly $m_i = 3$. If $\gamma = 0$ then $\phi(\Gamma) = \langle a \rangle$ which gives ϕ is not onto which contradicts our assumption that ϕ is onto, hence if all
$\phi(x_i) \in \langle b \rangle$ then for $m_i = 3$; $\gamma \ge 1$.

(iii) If all $\phi(x_i) \in \langle ab \rangle$ then $m_i = 2$ clearly $\gamma \ge 1$.

(iv) If some $\phi(x_i) \in \langle b \rangle$ for $i = 1,2, \ldots, s; s < k$ and remaining $(x_{s+j}) \in \langle b \rangle, j = 1,2, \ldots, t$. Then $s + t = k$, as s is even so k must be greater than equal 3 i.e.
$\gamma \ge 3$, furthermore

(a) If $k = 3$, then $s = 2$, $t = 1$, so that using [1] we get $\gamma \ge 0$.
(b) If $k > 3$, say $k = 4$ then $s = 2$, $t = 2$ using [1] we get $\gamma \ge 0$.
i.e. if $k \ge 3$ then $\gamma \ge 1$.

(v) If some $\phi(x_i) \in \langle a \rangle$ for $i = 1,2, \ldots, s; s < k$ and remaining
$(x_{s+j}) \in \langle ab \rangle, j = 1,2, \ldots, t$. Then $s + t = k$, as s and t both are even so k must be greater than equal 4 i.e. $k \ge 4$, furthermore

(i) If $k = 4$, then $s = 2$, $t = 2$, so that using [1] we get $\gamma \ge 0$.

(ii) If $k > 4$, say $k = 6$ then either $s = 4$, $t = 2$ or $s = 2$, $t = 4$; in both cases using [1] we get $\gamma \ge 0$.

i.e. if $k \ge 4$ then $\gamma \ge 0$.

(vi) If some $\phi(x_i) \in \langle b \rangle$ for $i = 1,2, \ldots, s; s < k$ and remaining
$(x_{s+j}) \in \langle ab \rangle$, $j = 1,2, \ldots, t$. Then $s + t = k$, as t is even so k must be greater than equal 3 i.e. $k \ge 3$, furthermore

(a) If $k = 3$, then $s = 1$, $t = 2$, so that using [1] we get $\gamma \ge 0$.
(b) If $k > 4$, say $k = 6$ then either $s = 4$, $t = 2$ or $s = 2$, $t = 4$; in both cases using [1] we get $\gamma \ge 0$.

i.e. if $k \geq 4$ then $\gamma \geq 0$.

(vii) If some $\phi(x_i) \in \langle a\rangle$ for $i = 1,2, \dots, s; s < k$ and some $(x_{s+j}) \in \langle b\rangle,\ j = 1,2, \dots . t,$ and remaining $(x_{s+t+j}) \in \langle ab\rangle,\ j = 1,2, \dots, p$. $s + t + p = k$. Then clearly k must be even and ≥ 6.

(i) If $k = 6$; $s = 2$, $t = 2$, $p = 2$ then from [1] $\gamma \geq 0$.

(ii) If $k \geq 6$ then also $\gamma \geq 0$.

Hence the condition is necessary for $k \neq 0$.

Next we see that the conditions are sufficient for $k \neq 0$.

2. (i) All $\phi(x_i) \in \langle a\rangle$, $mi = 4$, and $\gamma \geq 1$. Then let us construct ϕ as follows-

$\phi(x_i) = a^2$, $\phi(\alpha_1) = b^2$, $\phi(\beta_1) = b^{-k} b$

And $\phi(a_j) = 1 = \phi(b_j)$ $\forall j$ if any

Then $\prod_{i=1}^{k}\phi(x_i) \prod_{j=1}^{\gamma} [\phi(a_j), \phi(b_j)] = a^{2k}.[b^2, a^{-k}b]$

$$= a^{2k}.b^{-2}.(a^{-k}b)^{-1}.b^{-2}.a^{-k}b$$
$$= a^{2k}.b^{-2}.b^{-1}a^{k}.b^{2}.a^{-k}.b$$
$$= a^{2k}.b^{-3}.b^{-2}a^{-k}.a^{-k}.b$$
$$= a^{2k}.b^{-5}.a^{-2k}.b$$
$$= a^{2k}.a^{2k}.a^{5}.b$$
$$= a^{4k}.b^{3}.b^{3} \because a^4 = 1,\ b^3 = 1$$
$$=1$$

Also, $\langle b\rangle, \langle ab\rangle \in \phi(\Gamma)$ gives ϕ is a smooth epimorphism.

2 (ii) If all $\phi(x_i) \in \langle b\rangle$ clearly $m_i = 3$. $\gamma \geq 1$. Then let us construct ϕ as

$\phi(x_i) = b,\ \phi(\alpha_1) = a^2$, $\phi(\beta_1) = ab^{-2k}$

And $\phi(\alpha_j) = 1 = \phi(\beta_j)$ $\forall j$ if any

Then $\prod_{i=1}^{k}\phi(x_i) \prod_{j=1}^{\gamma} [\phi(\alpha_j), \phi(\beta_j)] = b^{k}.[a^2, ab^{-2k}]$

$$= b^{k}.a^{-2}.(ab^{-2k})^{-1}.a^{2}.ab^{-2k}$$
$$= b^{k}.a^{-2}.b^{2k}.a^{-1}.a^{2}.ab^{-2k}$$
$$= b^{k}.b^{-2k}.a^{2}a^{-1}.a^{2}.ab^{-2k}$$
$$= b^{-k}.a^{4}.b$$
$$= b^{-k}.1.b^{-2k},\ \because a^4 = 1$$
$$= b^{-3k} = 1 \quad \because b^3 = 1$$

Also, $\langle a\rangle, \langle ab\rangle \in \phi(\Gamma)$ gives ϕ is a smooth epimorphism.

2 (iii) If all $\phi(x_i) \in \langle ab\rangle$ then clearly $m_i = 2$ and $\gamma \geq 1$. Then let us construct ϕ as

$\phi(x_i) = a^{-2/k}\ \beta^2,\ \phi(\alpha_1) = a^3$, $\phi(\beta_1) = b^{2k}$

And $\phi(\alpha_j) = 1 = \phi(\beta_j)$ $\forall j$ if any

Then $\prod_{i=1}^{k}\phi(x_i) \prod_{j=1}^{\gamma} [\phi(\alpha_j), \phi(\beta_j)] = (a^{-2/k}b^2)^{k}.\ [a^3, b^{2k}]$

$= a^{-2}.b^{2k}.a^{-3}.b^{-2k}\ a^3.b^{3k}$
$= a^{-2}.a^3.b^{-2k}.b^{-4k}\ a^{-3}.$
$= a.b^{-6k}.a^{-3}.$
$= a.a^3.b^{6k}$
$= a^4.b^{3k}.b^{3k}$
$= 1$

Hence $\langle a\rangle, \langle b\rangle \in \phi(\Gamma)$ gives ϕ is a smooth epimorphism.

2 (iv) If some $\phi(x_i) \in \langle a\rangle$ for $i = 1,2, \dots ,s ; s < k$ remaining $(x_{s+j}) \in \langle b\rangle$, $j = 1,2, \dots , t$,

such that $s + t = k; k \geq 4$ then $\gamma \geq 0$.
(a) If $k = 4$ then $s = 2$, $t = 2$, $\gamma \geq 0$, then let us construct ϕ as
$\phi(x_1) = a,\ \phi(x_2) = a^3,\ \phi(x_3) = b,\ \phi(x_4) = b^2$
And $\phi(\alpha_j) = 1 = \phi(\beta_j)\ \forall j$ if any
Then $\prod_{i=1}^{4} \phi(x_i).\ 1 = a.a^3.b.b^2 = a^4.b^4 = 1$
(b) If $k \geq 4$, say $k = 6$, then $s = 4$, $t = 2$, $\gamma \geq 0$, then let us construct ϕ as
$\phi(x_1) = a,\ \phi(x_2) = a^2\ \phi(x_3) = a^3,\ \phi(x_4) = a^{-2},\ \phi(x_5) = b,\ \phi(x_6) = b^2$
And $\phi(\alpha_j) = 1 = \phi(\beta_j)\ \forall j$ if any
Then $\prod_{i=1}^{6} \phi(x_i).\ 1 = a.a^2.a^3.a^{-2}.b.b^2 = a^4.b^3 = 1$

Hence the condition is sufficient to yield ϕ is a smooth epimorphism.

2 (v) If some $\phi(x_i) \in \langle a\rangle$ for $i = 1,2, \dots ,s ; s < k$ and remaining $(x_{s+j}) \in \langle ab\rangle$, $j = 1,2, \dots , t$.
such that $s + t = k$; $k \geq 4$ then $\gamma \geq 0$.
(a) If $k = 4$ then $s = 2$, $t = 2$, $\gamma \geq 0$, then let us construct ϕ as
$\phi(x_1) = a^2,\ \phi(x_2) = a^3,\ \phi(x_3) = ab,\ \phi(x_4) = a^2b$
And $\phi(\alpha_j) = 1 = \phi(\beta_j)\ \forall j$ if any

$$\begin{aligned}\text{Then } \prod_{i=1}^{4} \phi(x_i).\ 1 &= a^2.a^3.ab.a^2b \\ &= a^6.a^{-2}.b^{-1}.\ b \\ &= a^4.\ 1 \\ &= 1\end{aligned}$$

(b) If $k \geq 4$, say $k = 6$, then $s = 2$, $t = 4$, $\gamma \geq 0$, then let us construct ϕ as
$\phi(x_1) = a^2,\ \phi(x_2) = a^3,\ \phi(x_3) = ab,\ \phi(x_4) = a^2b,$
$\phi(x_5) = a^2b^2,\ \phi(x_6) = b^2a^2$
And $\phi(\alpha_j) = 1 = \phi(\beta_j)\ \forall j$ if any

$$\begin{aligned}\text{Then } \prod_{i=1}^{6} \phi(x_i).\ 1 &= a^2.a^3.ab.a^2b.a^2\ b^2.b^2\ a^2 \\ &= a^6.a^{-2}.b^{-1}.b.b^{-2}.a^{-2}\ .\ a^{-2}b^{-2} \\ &= a^4.b^{-2}.1.b^2\end{aligned}$$

$= 1.\ \beta^{-2}.\ \beta^{2}$
$= 1$

Hence the condition that gives ϕ is a smooth epimorphism is sufficient.

2 (vi) If some $\phi(x_i) \in \langle b\rangle$ for $i = 1,2, \dots, s; s < k$ and remaining $\phi(x_{s+j}) \in \langle ab\rangle$, $j = 1,2, \dots, t$. such that $s + t = k$; $k \geq 4$ then $\gamma \geq 0$.

(a) If $k = 4$ then $s = 2$, $t = 2$, $\gamma \geq 0$, then let us construct ϕ as

$\phi(x_1) = b,\ \phi(x_2) = b^2,\ \phi(x_3) = ba^3,\ \phi(x_4) = ab^2$

And $\phi(\alpha_j) = 1 = \phi(\beta_j)\ \forall j$ if any

Then $\prod_{i=1}^{4} \phi(x_i).\ 1 = b.b^2.ba^3.ab^2$
$= b^4.a^4.b^2$
$= b^4.\ 1.\ b^2$
$= b^6$
$= b^3.\ b^3$
$= 1$

(b) If $k \geq 4$, say $k = 6$, then $s = 2$, $t = 4$, $\gamma \geq 0$, then let us construct ϕ as

$\phi(x_1) = b,\ \phi(x_2) = b^2,\ \phi(x_3) = ab,\ \phi(x_4) = a^2b,$

$\phi(x_5) = a^3b,\ \phi(x_6) = a^2b^2$

And $\phi(\alpha_j) = 1 = \phi(\beta_j)\ \forall j$ if any

Then $\prod_{k=1}^{6} \phi(x_i).\ 1 = b.b^2.ab.a^2b.a^3b.a^2\,b^2$
$= b^3.a.a^{-2}.b^{-1}.b.a^3\,.b.\ a^2.b^2$
$= 1.a^{-1}.1.a^3.a^{-2}.b^{-1}.b$
$= a^{-1}.\ a.\ 1$
$= 1$

Hence the condition that gives ϕ is a smooth epimorphism is sufficient.

(viii) If some $\phi(x_i) \in \langle a\rangle$ for $i = 1,2, \dots, s; s < k$ and some $\phi(x_{s+j}) \in \langle b\rangle$, $j = 1,2, \dots, t$. and remaining $(x_{s+t+j}) \in \langle ab\rangle$, $j = 1,2, \dots, p$. $s + t + p = k$. Then clearly k must be even and ≥ 6.

(a) If $k = 6$; $s = 2$, $t = 2$, $p = 2$, $\gamma \geq 0$. then let us construct ϕ as

$\phi(x_1) = a^2,\ \phi(x_2) = a^3,$

$\phi(x_3) = b,\ \phi(x_4) = b^3,$

$\phi(x_5) = a^2b^2,\ \phi(x_6) = a^3b,$

And $\phi(\alpha_j) = 1 = \phi(\beta_j)\ \forall j$ if any

Then $\prod_{k=1}^{6} \phi(x_i).\ 1 = a^2.a^3.b.b^2.a^2b^2.\ a^3\,b$
$= a^5.b^3.b^{-2}.a^{-2}.a^3\ b$
$= a.1.b^{-2}.ab$

$= a.a^{-1}b^2 . b$
$= a^4.b^3 = 1$
$= 1$

(c) If $k \geq 6$, say $k = 8$, then $s = 2$, $t = 4$, $p = 2$, $\gamma \geq 0$,
Then let us construct ϕ as
$\phi(x_1) = a^2$, $\phi(x_2) = a^3$, $\phi(x_3) = b$, $\phi(x_4) = b^2$, $\phi(x_5) = b^{-1}$, $\phi(x_6) = b^{-2}$, $\phi(x_7) = (a^3b^2a)$,
$\phi(x_8) = (ba^3b)$,
And $\phi(\alpha_j) = 1 = \phi(\beta_j)\ \forall j$ if any

Then $\prod_{k=1}^{6} \phi(x_i). 1 = a^2.a^3.b.b^2 . b^{-1}.b^{-2} . (a^3.b^2 a). (ba^3b)$
$= a^5.b^3.b^{-3}.a^3.a^{-1}.b^{-2} .b.a^3.b$
$= a.1.1.a^2.a^{-3}.1$
$= a.a^{-1}$
$= 1$

Hence $\langle a\rangle, \langle b\rangle, \langle ab\rangle \in \phi(\Gamma)$ implies ϕ is a smooth epimorphism.
This proves the sufficiency of the conditions.

Conclusion

From this above discussion we can establish a set of necessary and sufficient condition for the existence of a smooth epimorphism ϕ from Γ to $G(CCl_4)$, and hence we have the smooth quotient

$$\frac{\Gamma}{ker\phi} \cong G(CCl_4).$$

Clearly $ker\phi$ is a surface group of genus g (≥ 2).

Future prospect: The study of upper bound for the order of the group of automorphisms of compact Riemann surface with reference to the point group of carbon tetrachloride molecule is our future project in this topic, which we are going to be published very soon.

Acknowledgements

The author would like to thank the referee of the paper[13] for the valuable remark that encouraged the writing of the current paper.

- This project did not receive any specific grant from any funding agencies in the public, commercial or not-for-profit sectors.

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