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Disjoint total Restrained Dominating sets in Graphs

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Abstract

The disjoint total restrained domination number of a graph G is the minimum cardinality of the union of two disjoint total restrained dominating sets in G. We also consider an invariant the minimum cardinality of the disjoint union of a restrained dominating set and a total restrained dominating set. In this paper, we initiate a study of these parameters and establish some results on these new parameters.

Key words: Inverse domination set, restrained dominating set, total restrained dominating set, inverse total restrained dominating set, disjoint total restrained domination number.

AMS Subject Classification: 05*C*69, 05*C*78

1. Introduction

We consider a graph $G = (V, E)$ with vertex set V and edge set E which is finite, undirected without loops and multiple edges, Any undefined term here may be found in Kulli[1, 2]. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Recently many domination parameters are given in the books by Kulli in[2,3,4].

Let D be a minimum dominating set of G. If $V - D$ contains a dominating set D' of G, then D. is called an inverse dominating set of G with respect to D. The inverse domination number $\gamma^{-1}(G)$ of G is the minimum cardinality of an inverse dominating set of G. This concept was introduced by Kulli *et al*. in[5]. Many other inverse domination parameters were studied, for example, in[8, 9, 10, 12, 14, 15, 17].

A dominating set D in G is a restrained dominating set if the induced sub graph $\langle V - D\rangle$ has no isolated vertices. Alternately, a set $D \subseteq V$ is a restrained dominating set if every vertex not in D is adjacent to a vertex in D and to a vertex in $V - D$. This concept was studied by Domke *et al*. in[6] and was also studied as cototal domination in graphs by Kulli *et al*. in[7].

The disjoint restrained domination number $\gamma_r\gamma_r(G)$ of G is defined as follows: $\gamma_r\gamma_r(G)=\min\{|D_1|+|D_2| : D_1, D_2$ are disjoint restrained dominating sets of $G\}$. This concept was introduced by Kulli in[8]. Many other disjoint domination parameters were studied, for example, in[9, 10, 11, 12, 13, 14, 15].

A set $D \subseteq V$ is a total restrained dominating set if every vertex of G is adjacent to a vertex in D and every vertex of $V - D$ is adjacent to a vertex in $V - D$. The total restrained domination number $\gamma_{tr}(G)$ of G is the minimum cardinality of a total restrained dominating set of G, see[16]. Let D be a minimum total restrained dominating set of G. If $V - D$ contains a total restrained dominating set D' of G, then D' is called an inverse total restrained dominating set with respect to D. The inverse total restrained domination number $\gamma_{tr}^{-1}(G)$ of G is the minimum cardinality of an inverse total restrained dominating set of G. The concept was introduced by Kulli in[17].

Two graphs G_1 and G_2 have disjoint vertex sets V_1, V_2 and edge sets E_1, E_2 respectively. Their union is defined by $G_1 \cup G_2$ and it has $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Their join is denoted by $G_1 + G_2$ and it consists of $G_1 \cup G_2$ and all edges joining every vertex of V_1 with every vertex of V_2. The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where i^{th} vertex G_1 is adjacent to every in the i^{th} copy of G_2.

In this paper, we initiate a study of the disjoint total restrained domination number of a graph.

2. Disjoint Total Restrained Domination :

We now introduce the concept of disjoint total restrained domination number in graphs.

Definition 1. The disjoint total restrained domination number $\gamma_{tr}\gamma_{tr}(G)$ of a graph G is defined as follows: $\gamma_{tr}\gamma_{tr}(G) = \min\{|D_1|+|D_2| : D_1, D_2$ are disjoint total restrained dominating sets of $G\}$.

Remark 2. Not all graphs have a disjoint total restrained domination number. For Example, the cycle C_5 does not have two disjoint total restrained dominating sets.

Theorem 3. γ_{tr}^{-1}-set exists in a graph G with p vertices, then

$$2\gamma_{tr}(G) \le \gamma_{tr}\gamma_{tr}(G) \le \gamma_{tr}(G) + \gamma_{tr}^{-1}(G) \le p.$$

We also consider an invariant the minimum cardinality of the disjoint union of a retrained dominating set D and a total restrained dominating set D' and it is denoted by $\gamma_r\gamma_{tr}(G)$. We call such a pair of dominating sets (D, D'), a $\gamma_r\gamma_{tr}$-pair. A $\gamma_r\gamma_{tr}$-pair can be found by letting D' be any total restrained dominating set, and then noting that the complement $V - D'$ contains a minimal restrained dominating set D.

Remark 4. Not all graphs have a $\gamma_r\gamma_{tr}$-pair. For example the cycle C_5 does not have a $\gamma_r\gamma_{tr}$-pair.

Proposition 5. If both $\gamma_r\gamma_{tr}$-pair and $\gamma_{tr}\gamma_{tr}$-pair exist, then

$$\gamma_r\gamma_{tr}(G) \le \gamma_{tr}\gamma_{tr}(G).$$

Proposition A[16]

(1) $\gamma_{tr}(K_p) = 2$ if $p \geq 4$.

(2) $\gamma_{tr}(K_{m,n}) = 2$, if $2 \leq m \leq n$.

(3) $\gamma_{tr}(C_p) = p - 2\left\lfloor \frac{p}{4} \right\rfloor$, if $p \geq 4$.

Proposition B[17]. For a complete graph K_p, $p\geq 4$, $\gamma_{tr}^{-1}(K_p) = 2$.

Proposition 6. For any $p\geq 4$,

$$2\gamma_{tr}(K_p) = \gamma_{tr}\gamma_{tr}(K_p) = 4.$$

Proposition 7. For any $4 \leq m \leq n$,

$$2\gamma_{tr}(K_{m,n}) = \gamma_{tr}\gamma_{tr}(K_{m,n}) = 4.$$

Definition 8. A graph G is called $\gamma_{tr}\gamma_{tr}$-minimum if $\gamma_{tr}\gamma_{tr}(G) = 2\gamma_{tr}(G)$.

Definition 9. A graph G is called $\gamma_{tr}\gamma_{tr}$-maximum if $\gamma_{tr}\gamma_{tr}(G) = p$.

The following classes of graphs are $\gamma_{tr}\gamma_{tr}$-minimum.

1) The complete graphs K_p, $p\geq 4$, are $\gamma_{tr}\gamma_{tr}$-minimum.
2) The complete bipartite graphs $K_{m,n}$, $4 \leq m \leq n$, are $\gamma_{tr}\gamma_{tr}$-minimum.
3) All cycles C_{4n}, $n \geq 1$, are $\gamma_{tr}\gamma_{tr}$-minimum.

The following classes of graphs are $\gamma_{tr}\gamma_{tr}$-maximum.

1) The complete graphs K_4 is $\gamma_{tr}\gamma_{tr}$-maximum.
2) All cycles C_{4n}, $n \geq 1$, are $\gamma_{tr}\gamma_{tr}$-maximum.

Theorem 9. For each even integer $p\geq 4$, there exists a connected graph G such that $\gamma_{tr}^{-1}(G) - \gamma_{tr}(G) = p - 4$ and $\gamma_{tr}(G) + \gamma_{tr}^{-1}(G) = p$.

Proof: Let $p\geq 4$ be an even integer. Consider the graph G with p vertices as in Figure 1.

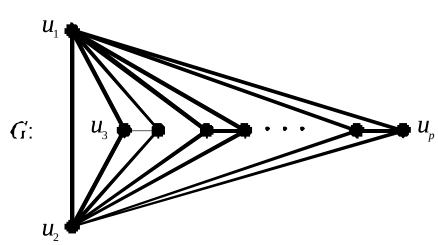

Figure 1

Then $D = \{u_1, u_2\}$ is a total restrained dominating set in G, which is minimum. Thus $\gamma_{tr}(G) = 2$. Since u_3u_4, u_5u_6, ... , $u_{p-1}u_p$ are edges in G, it implies that $D_1 = V(G) - \{u_1, u_2\}$ is a minimum total restrained dominating set in $V - D$. Thus $\gamma_{tr}^{-1}(G) \leq |D_1| = p - 2$. Since $N_G[S] \neq V(G)$ for all proper subsets of S of D_1, it implies that $\gamma_{tr}^{-1}(G) = |D_1| = p - 2$. Thus $\gamma_{tr}^{-1}(G) - \gamma_{tr}(G) = p - 4$ and also $\gamma_{tr}(G) + \gamma_{tr}^{-1}(G) = p$.

Corollary 10. The difference $\gamma_r^{-1}(G) - \gamma_r(G)$ can be made arbitrarily large.

Theorem 11. For each integer $n \geq 1$, there exists a connected graph G such that $\gamma_{tr}(G) + \gamma_{tr}^{-1}(G) - \gamma_{tr}\gamma_{tr}(G) = 2n$.

Proof: Consider the graph G as in Figure 2 obtained by adding to the corona $C_4 \circ C_4$ $2n$ vertices $x_1, y_1, x_2, y_2, ..., x_n, y_n$ and the edges x_iu_j, y_iu_j, x_iy_i, $i = 1,2, \ldots, n$, $j = 1, 2, 3, 4$.

G:

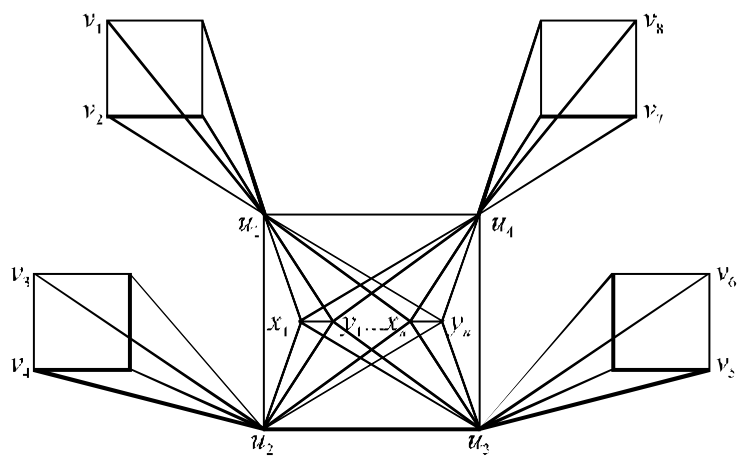

Figure 2

Then $\{u_1, u_2, u_3, u_4\}$ is the unique minimum total restrained dominating set in G and $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \cup \{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ is a γ_{tr}^{-1}-set in G. Thus $\gamma_{tr}(G) = 4$ and $\gamma_{tr}^{-1}(G)=8 + 2n$. Also the sets $D_1=\{u_1, u_2, v_5, v_6, v_7, v_8\}$ and $D_2=\{u_3, u_4, v_1, v_2, v_3, v_4\}$ constitute a $\gamma_{tr}\gamma_{tr}$-pair in G. Hence $\gamma_{tr}\gamma_{tr}(G) = |D_1|+|D_2| = 12$.

Thus $\gamma_r(G) + \gamma_r^{-1}(G) - \gamma_r\gamma_r(G) = 2n$.

Corollary 12. The difference $\gamma_{tr}(G) + \gamma_{tr}^{-1}(G) - \gamma_{tr}\gamma_{tr}(G)$ can be made arbitrarily large.

Theorem 13. Let G and H be nontrivial complete graphs, then

$$2\gamma_{tr}(G+H) = \gamma_{tr}\gamma_{tr}(G+H) = \gamma_{tr}(G + H) + \gamma_{tr}^{-1}(G+H) = 4.$$

Proof: Let G and H be complete graphs with $|V(G)| \geq 2$ and $|V(H)| \geq 2$. In $G + H$, each vertex of G is adjacent to every vertex of H and vice versa. Hence $G+H$ is a complete graph with at least 4 vertices. Thus by Proposition A(1), $\gamma_{tr}(G+H)=2$ and by Proposition B, $\gamma_{tr}^{-1}(G+H) = 2$. Thus

$$2\gamma_{tr}(G+H) = \gamma_{tr}\gamma_{tr}(G+H) = \gamma_{tr}(G + H) + \gamma_{tr}^{-1}(G+H) = 4.$$

Corollary 14. Let G and H be nontrivial complete graphs and $|V(G + H)| = p$.
Then $\gamma_{tr}(G + H) + \gamma_{tr}^{-1}(G+H) = p$ if and only if $G=K_2$ and $H=K_2$.

3. Some Open Problems

Problem 1. Characterize graphs G for which $\gamma_r\gamma_{tr}(G) = \gamma_{tr}\gamma_{tr}(G)$.

Problem 2. Characterize the class of $\gamma_{tr}\gamma_{tr}$ -minimum graphs.

Problem 3. Characterize the class of $\gamma_{tr}\gamma_{tr}$ -maximum graphs.

Problem 4. Obtain bounds for $\gamma_{tr}\gamma_{tr}(G)+\gamma_{tr}\gamma_{tr}(\overline{G})$.

Problem 5. What is the complexity of the decision problem corresponding to $\gamma_{tr}\gamma_{tr}(G)$?

Problem 6. Is DISJOINT TOTAL RESTRAINED DOMINATION NP-complete for a class of graphs?

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