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## Advances and Required Changes in Topology Discovered in the Continued Investigation of $T_0$ -identification Spaces

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**Abstract**

In this paper discoveries and additional, useful topological tools revealed in the continued investigation of  $T_0$ -identification spaces are given, and easily used conditions that imply a space is not completely regular or  $T_{3(1)2}$  are given.

*Key words* :  $T_0$ -identification spaces, classical separation axioms, negations.

**Subject classification**: 54D10, 54D15.

**1. Introduction and Preliminaries**

In the late 1980's and early 1990's great progress was being made in the development and expansion of mathematical knowledge, but there was an obstacle hindering its continued growth and expansion. There was no common mathematical language or common classification used by the different groups of creative mathematicians making communication between the groups extremely difficulties, if not impossible.

To overcome the difficulties in communication, a group of mathematicians joined together to develop a common language and common classification to be used by all within mathematics. To promote and motivate the use of their common language and common classification, they eventually and cleverly created an abstract model that included the mathematics done in that time period as a special case. The development and investigation of that new model led to a branch of mathematics today called modern day topology.

Within mathematics, the ability to separated distinct elements, closed sets and elements not in the closed set, and disjoint closed sets is important and was included in the new mathematical model and eventually were jointly referred to as separation axioms. Those separation axioms include  $T_0$ ,  $T_1$ ,  $T_2$ , Urysohn, regular,  $T_3$ , completely regular,  $T_{3(1)2}$ , normal, and  $T_4$ , which are today commonly referred to as classical separation axioms.

*Definition 1.1.* A topological space  $(X,T)$  is  $T_0$  iff for distinct elements  $x$  and  $y$  in  $X$  there exists an open set containing only one of  $x$  and  $y$ .

All spaces in the paper are topological spaces.

*Definition 1.2.* A space  $(X,T)$  is  $T_1$  iff for distinct elements  $x$  and  $y$  in  $X$  there exists an open set containing  $x$  and not  $y$  and an open set containing  $y$  and not  $x$ .

*Definition 1.3.* A space is  $T_2$  iff for distinct elements  $x$  and  $y$  in  $X$  there exist two disjoint open sets with  $x$  in one and  $y$  in the other.

*Definition 1.4.* A space is Urysohn iff for distinct elements  $x$  and  $y$  in  $X$  there exist open sets  $U$  and  $V$  such that  $x$  is in  $U$ ,  $y$  is in  $V$ , and  $Cl(U)$  and  $Cl(V)$  are disjoint.

*Definition 1.5.* A space  $(X,T)$  is regular iff for each closed set  $C$  and each element  $x$  not in  $C$  there exist two disjoint open sets one containing  $x$  and the other containing  $C$ . A regular  $T_1$  space is denoted by  $T_3$ .

*Definition 1.6.* A space  $(X,T)$  is completely regular iff for each closed set  $C$  and each  $x$  not in  $C$  there exists a continuous function  $f$  from  $(X,T)$  onto  $([0,1],U)$  such that  $f(x) = 0$  and  $f(C) = 1$ , where  $U$  is the relative absolute value metric topology on  $[0,1]$ . A completely regular  $T_1$  space is denoted by  $T_{(3(1)2)}$ .

*Definition 1.7.* A space  $(X,T)$  is normal iff for disjoint closed sets  $C$  and  $D$  there exist two disjoint open sets one containing  $C$  and the other containing  $D$ . A normal  $T_1$  space is denoted by  $T_4$ .

The work of the 1990's topological study pioneers was fruitful and the language and classification introduced by them became the language and classifications used in mathematics. Fortunately for the future mathematicians, the old masters left some very logical, natural questions unaddressed in their work.

As seen above, in Definition 1.5, Definition 1.6, and Definition 1.7, two separation axioms that differed only by the  $T_1$  separation axiom were given, unlike the earlier definitions. Thus two very logical, natural questions arise: (1) Could  $T_1$  in each definition be replaced by the weaker  $T_0$  separation axiom? and, if so, (2) Are there separation axioms weaker than  $T_i$ ,  $i = 0,1,2$ , and Urysohn, which together with  $T_0$ , equals  $T_i$ ,  $i = 0,1,2$ , or Urysohn, respectively?

Also, of the separation axioms given above, completely regular and  $T_{(3(1)2)}$  are unique in that a continuous function satisfying strict conditions for each closed set  $C$  and each element not in  $C$  is required. Thus knowing a space is completely regular or  $T_{(3(1)2)}$  gives a strong property with which to work. However, not knowing if a space is completely regular or  $T_{(3(1)2)}$  and wanting to know if the space is completely regular or  $T_{(3(1)2)}$  can be challenging, raising the question of whether there are easily used properties that can be checked to show that the space is neither completely regular nor  $T_{(3(1)2)}$ .

Below, the questions above are addressed.

## 2. Answers to Basic, logical Unanswered Questions.

The investigation of question (1) has shown that for regular and completely regular,  $T_1$  can be replaced by  $T_0$ , but not for normal. Thus, the normal separation axiom differs from regular and completely regular and raises the question: For what property  $P$ , if any, would  $((\text{normal and } P) \text{ and } T_0) = T_4$ ?

A partial solution to question (2) was given in 1961 [1]: A space is  $T_2$  iff it is  $(R_1 \text{ and } T_0)$  and a space is

$T_1$  iff it is  $(R_0$  and  $T_0)$ .

The  $R_1$  separation axiom was introduced in the 1961 paper<sup>1</sup> and the  $R_0$  separation axiom was introduced in 1943<sup>13</sup>. A space  $(X,T)$  is  $R_1$  iff for  $x$  and  $y$  in  $X$  such that  $Cl(\{x\})$  is unequal to  $Cl(\{y\})$ , there exist disjoint open sets, one containing  $x$  and the other containing  $y$ . A space  $(X,T)$  is  $R_0$  iff for each closed set  $C$  and each  $x$  not in  $C$ ,  $C$  and  $Cl(\{x\})$  are disjoint.

To resolve the question concerning  $T_0$  a new, never before even imagined topological property discovered in a 2017 paper<sup>2</sup> was needed:  $L = (T_0$  or “not- $T_0$ ) =  $(P$  or “not- $P$ )”, where  $P$  is a topological property for which “not- $P$ ” exists, is the least of all topological properties<sup>2</sup> and a space is  $T_0$  iff it is  $(L$  and  $T_0)$ <sup>3</sup>.

In a 1988 paper<sup>4</sup>, it was shown that a space is Urysohn iff it is  $(($ weakly Urysohn) and  $T_0)$ . The (weakly Urysohn) separation axiom was introduced in the 1988 paper<sup>4</sup>: A space  $(X,T)$  is (weakly Urysohn) iff for  $x$  and  $y$  in  $X$  such that  $Cl(\{x\})$  is not equal to  $Cl(\{y\})$ , there exist two open sets, one containing  $x$  and the other containing  $y$ , whose closures are disjoint.

In a 2017 paper<sup>5</sup>, the question concerning normal was resolved: A space is  $T_4$  iff it is  $(($ normal and  $R_0)$  and  $T_0)$ .

The discovery of  $L$ , the least topological property, raised the question of whether for each classical separation axiom given above, is there is a least topological property, which together with  $T_0$ , equals the classical separation axiom?, which is addressed below.

### 3. Least Properties and Properties that Show a Space is not Completely Regular or $T_{(3(1)2)}$ .

The never before imagined topological property used to resolve the question above about  $T_0$  was discovered in the continued study of  $T_0$ -identification spaces introduced in 1936<sup>14</sup>.

*Definition 3.1.* Let  $(X,T)$  be a space, let  $R$  be the equivalence relation on  $X$  defined by  $xRy$  iff  $Cl(\{x\}) = Cl(\{y\})$ , for each  $x$  in  $X$ , let  $C_x$  be the  $R$  equivalence class containing  $x$ , let  $X_0$  be the set of  $R$  equivalence classes, let  $N$  be the natural function from  $X$  onto  $X_0$ , and let  $Q(X,T)$  be the decomposition topology on  $X_0$  determined by  $N$  and  $(X,T)$ . Then  $(X_0, Q(X,T))$  is the  $T_0$ -identification space of  $(X,T)$ .

$T_0$ -identification spaces were cleverly created to give for each space  $(X,T)$  a closely  $(X,T)$  related  $T_0$ -identification space with the  $T_0$  axiom added, making  $T_0$ -identification spaces a useful topological tool<sup>14</sup>. In the 1936 paper,  $T_0$ -identification spaces were used to jointly characterize pseudometrizable and metrizable: A space is pseudometrizable iff its  $T_0$ -identification space is metrizable.

In 1975<sup>12</sup>,  $T_0$ -identification spaces were used to jointly characterize  $R_1$  and  $T_2$ : A space is  $R_1$  iff its  $T_0$ -identification space is  $T_2$ .

In 2015<sup>6</sup>, using the characterizations of pseudometrizable and  $R_1$  as both motivation and model, weakly  $P_0$  spaces and properties were introduced.

*Definition 3.1.* Let  $P$  be a topological property for which  $P_0 = (P$  and  $T_0)$  exists. Then a space  $(X,T)$  is weakly  $P_0$  iff its  $T_0$ -identification space  $(X_0, Q(X,T))$  has property  $P$ . A topological property  $P_0$  for which weakly  $P_0$  exists is called a weakly  $P_0$  property.

Because of the special  $T_0$  property for  $T_0$ -identification spaces given above, a space is weakly  $P_0$  iff its  $T_0$ -identification space is  $P_0$ . Thus, by the results above, pseudometrizable = weakly (pseudometrizable)<sub>0</sub> = weakly metrizable and  $R_1 =$  weakly  $(R_1)_0 =$  weakly  $T_2$ .

The continued investigation of weakly Po spaces and properties revealed that  $R_0 = \text{weakly } (R_0)_o = \text{weakly } T_1$ <sup>6</sup>, (weakly Urysohn) = weakly (Urysohn)<sub>o</sub> = weakly Urysohn<sup>7</sup>, regular = weakly (regular)<sub>o</sub> = weakly  $T_3$ <sup>7</sup>, completely regular = weakly (completely regular)<sub>o</sub> = weakly  $T_{(3(1)2)}$ <sup>7</sup>, normal = weakly (normal)<sub>o</sub><sup>5</sup>, and (normal and  $R_0$ ) = weakly (normal and  $R_0$ )<sub>o</sub> = weakly  $T_4$ <sup>5</sup>.

In the 2015 paper<sup>6</sup>, the search for a topological property that is not weakly Po led to the need and use of “not- $T_0$ ” revealing “not- $T_0$ ” as an important topological property for both additional study and use, leading to the inclusion of “not- $T_0$ ” and “not-P”, where P is a topological property for which ‘not-P’ exists, into the study of topology opening the path to topologically rich, not before imagined territory within topology leading to the discovery of L and other properties that have changed to study of topology forever.

As expected, the existence of the never before least topological property L created discontinuities in the study of topology. Using the 1930 definition of product properties, L is a product property creating a discontinuity in the study of product properties<sup>2</sup>. The discontinuity was corrected by the removal of L as a product property<sup>2</sup>. In addition, within classical topology, the question of whether (P and Q) is a product property for product properties P and Q was never addressed. The investigation of that question revealed L as the only topological property P for which “not-P” does not exist<sup>3</sup>, which was used to prove that for product properties P and Q, (P and Q) is a product space property<sup>8</sup>. Thus many new product properties were given and the use of “not-P”, where P is a product property, was used to give many new examples of not product properties<sup>8</sup>. In a similar manner, L created a discontinuity in the study of subspace properties<sup>9</sup>, which was resolved in a similar manner correcting and expanding the knowledge of subspace properties.

Initially, the search for properties that are weakly Po was trial and error, with no certainty of success, but a major, never before imagined breakthrough was given in a 2017 paper<sup>10</sup>: {Q<sub>o</sub>| Q is a topological property and weakly Q<sub>o</sub> exists} = {Q<sub>o</sub>| Q is a topological property and Q<sub>o</sub> exists}. The continued investigation of the question concerning least topological properties given above led to the following result: For a topological property Q for which Q<sub>o</sub> exists, the least topological property, which together with  $T_0$ , equals Q<sub>o</sub> is ((weakly Q<sub>o</sub>) or “not- $T_0$ ”)<sup>3</sup>. Thus, the least topological property, which together with  $T_0$ , equals  $T_0$  is (L or “not- $T_0$ ”), the least topological property, which together with  $T_0$ , equals  $T_1$  is ( $R_0$  or “not- $T_0$ ”), the least topological property, which together with  $T_0$ , equals  $T_2$  is ( $R_1$  or “not- $T_0$ ”), the least topological property, which together with  $T_0$ , equals Urysohn is ((weakly Urysohn) or “not- $T_0$ ”), the least topological property, which together with  $T_0$ , equals  $T_3$  is (regular or “not- $T_0$ ”), the least topological property, which together with  $T_0$ , equals  $T_{(3(1)2)}$  is ((completely regular) or “not- $T_0$ ”), the least topological property, which together with  $T_0$ , equals (normal and  $T_0$ ) is (normal or “not- $T_0$ ”), and the least topological property, which together with  $T_0$ , equals  $T_4$  is ((normal and  $R_0$ ) or “not- $T_0$ ”). Also, the least topological property, which together with  $T_0$ , equals metrizable is (pseudometrizable or “not- $T_0$ ”).

Thus, as stated above, the addition and use of “not- $T_0$ ” to the study of topology has provided the tool needed to unlocked many very natural, previously unaddressed questions. Likewise, the addition and use of “not-P”, where P is a topological property for which “not-P” exists, has proven to be a useful tool in the study of topology. As given above L is the only topological property P for which “not-P” does not exist, which was used to prove there is no strongest topological property<sup>11</sup>. Also, as given below, certain topological properties “not-P”, if satisfied by a space, can be used to show that the space is not completely regular or  $T_{(3(1)2)}$ .

Since completely regular implies regular, which implies (weakly Urysohn), which implies  $R_1$ , which implies  $R_0$ , then, by use of contrapositive statements, “not- $R_0$ ” implies “not- $R_1$ ”, which implies “not-(weakly Urysohn)”, which implies “not-regular”, which implies “not-completely regular”. Thus, given a space and

wanting to know if the space is completely regular, the space could be easily checked to see if it is “not- $R_0$ ” and, if so, the space is “not-completely regular” and thus not completely regular. In like manner, each of “not- $R_1$ ”, “not-(weakly Urysohn)”, and “not-regular” could be checked to show that a space is not completely regular.

Since  $T_{(3(12))}$  implies  $T_3$ , which implies Urysohn, which implies  $T_2$ , which implies  $T_1$ , which implies  $T_0$ , then, by use of contrapositive statements, “not- $T_0$ ” implies “not- $T_1$ ”, which implies “not- $T_2$ ”, which implies “not-Urysohn”, which implies “not- $T_3$ ”, which implies  $T_{(3(12))}$ . Thus, as above, if a space satisfies any of “not- $T_0$ ”, “not- $T_1$ ”, “not- $T_2$ ”, “not-Urysohn”, or “not- $T_3$ ”, then the space is not  $T_{(3(12))}$ .

Thus, the continued investigation of  $T_0$ -identification spaces through weakly  $P_0$  spaces and properties has been, and continues to be, a productive, enlightening, corrective study adding needed fundamental, foundational knowledge to the study of topology.

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