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A study on Strongly U-Flat Modules over Matlis Domains

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Abstract

In thbe article, R be a ring with local units that have discussed. For any ME mod-R, the map $\mu(M_1+M_2)$: $(M_1+M_2) \rightarrow (M_1+M_2)$ given by $\sum_{\underset{j=1}{j-1}}^{n}(Mi + Mj) \otimes r_i \rightarrow \sum_{\underset{j=1}{j-1}}^{n}(Mi + Mj) \otimes r$ be an isomorphism of right R-modules. Strongly U-Flat Modules over Matlis Domains has defined and discussed with their properties to know the relations.

Key words: U-flat modules, Matlis domains, Prufer domain, Matlis cotorsion.

Mathematics subject classification: Primary; 13C11, 13C13 Secondry: 13D07, 13G05.

1. Introduction

"**A** vector space M over a field R be a set of objects called vectors, which may be added, subtracted and multiplied by scalars (members of the underlying field). Thus M be an Abelian group under addition, and for every $r \in$ Rand $x \in M$ we have an element $rx \in M$. Scalar multiplication is distributive and associative. The multiplicative identity of the field acts as an identity on vectors. Consider a ring R with a local is U-flat as a left R-module"[6].

1. $A= (A_1 +A_2)$ be a pure ideal of R.
2. For every finite family $\alpha_i \leq i \leq n$ of the element of A, there exists $t \in A$ such that $\alpha_i = \alpha_i t \ \forall \ i \leq i \leq n$.
3. For all $\alpha \in$ A there exists $\beta \in A$ such that $\alpha \in \alpha\beta$.

4. $\frac{R}{(A1 + A2)}$, be a w-flat R-module.

Moreover, A is finitely generated, and then A is pure if and only if it is generated by an idempotent. Let O→A B→ C→O be an exact sequence such that A and C are strongly U- Flat modules, then B is strongly U- Flat modules. Let R be a semi- Dedekind domain. If M be a w-projective R-module and N be a weak U-projective R-module then $(M_1+M_2) \otimes_R^N$ is weak U-projective. The subsequent statements are equivalent,[1,17]

1. "R is the Prufer domain.
2. Every R-module is pure U-projective.
3. Ext_R^I $(M_1+M_2, N)=0$ for all pure U-projective R-module N.
4. Every pure U-projective R-module has on the injective envelope with the unique mapping property"[1].

If M be an R-module, then the subsequent are equivalent[4]:

1. "M is pure U-projective; Where $M= M_1 +M_2$.
2. M is pure projective concerning every exact sequence O→A→B→O, where A is pure U-projective.
3. For every exact sequence 0→ K→ F→ M→ *0* with ker t = O where F is pure W-injective, K→ F be a pure U-injective pre-envelope of K.
4. M is co-kernel of a pure U-injective pre-envelope of K→F with F projective"[4].

Let R be a semi-Dedekind domain, for an R-module M; the subsequent statements are equivalent[17];

1. "P wid (M) ≤ n.
2. Ext_R^{R+1} (N, M) = O for all R-module N of U-dimension ≤ 1.
3. If the sequence $0\to M \to E_0 \to E_1\to\ldots\ldots\to E_n \to O$ is exact with E_0, $E_1 = E_{n=1}$ pure U-injective, then also E_n is pure U-injective"[17].

Let R be a semi-Dedekind domain, for an R-module M and an integer n ≥ 0 the subsequent statements are equivalent[17];

1. "P wid (M) ≤ n.
2. Ext_R^{n+i} (N, M) = O for any pure U –injective R-module N.
3. Ext_R^{n+j} (M, N) = O for any pure U-injective R-module N and j ≥1.
4. There exists an exact sequence, $0\to P_0 \to P_{n+1}\to\ldots\ldots\to P_1\ \ P_0 \to M \to O$ where every P_i is pure U-projective"[17].

Definition 1.1: "Let R be an arbitrary ring, we will say that a module M ∈ R- mod is U- flat if the function- $\otimes_R(M_1 + M_2)$ is exact on the category R-module. In other words, if whenever $O\to A\otimes_R (M_1 + M_2) \to B \otimes_R(M_1 + M_2) \to C \otimes_R(M_1 + M_2) \to O$ be a short exact sequence. Since $O\to\otimes_R(M_1, M_2)$ is exact on the R.H.S., the module M is W- flat if any $A \otimes_R(M_1 + M_2)\to B \otimes_R(M_1 + M_2)$"[17].

Lemma 1.2: Let R be a ring with local units. For any ME mod-R the map, $\mu (M_1 + M_2) : (M_1 + M_2)\to (M_1 + M_2)$ given by $\sum_{i-1, j=1}^{n}(M_i + M_j) \otimes r_j \longrightarrow \sum_{i-1, j=1}^{n}(M_i + M_j) \otimes r$ be an isomorphism of right R-modules[6].

Proof: Since $(M_1 + M_2)$ R = $(M_1 + M_2)$ this map is clearly an epimorphism. Suppose $\sum_{i-1, j=1}^{n}(M_i + M_j)\ r_i = O$. Let $\in$ be a local unit in ring R satisfying $er_i = r_i e = r_i$ for i =1, 2, 3 ……….. n. Then,

$$\sum_{i-1, j=1}^{n}(M_i + M_j)\otimes r_i = \sum_{i-1, j=1}^{n}(M_i + M_j)\otimes r_i \in = \sum_{i-1, j=1}^{n}(m_i r_i + m_j r_j)\otimes \in = \otimes \in = 0.$$

Corollary 1.3: A ring R with local units is U-flat as a left R-module[4,17].

Proof: Let $A = A_1 + A_2$ and $B = B_1 + B_2$, let $O \to A \to B$ be an exact sequence of right R-modules. Tensoring with the left R-module R, This leads to a commutative diagram:

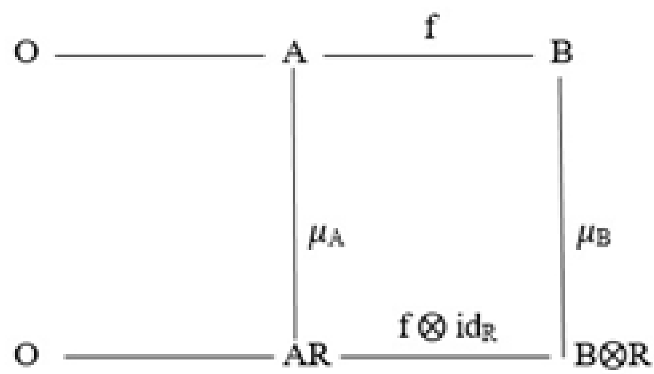

Figure 1.1: Diagram to explain commutative property[4,17].

Here id_R is the identity map on R and μ_A, μ_B are the isomorphism defined in lemma 1. Since μ_A, f and μ_B are all monomorphisms, so is $f \otimes id_R$. Hence R is U- flat in R-Mod[4,17].

Proposition 1.4: Let A be an ideal of a ring R. The subsequent conditions are equivalent[23];

1. "$A= (A_1 + A_2)$ be a pure ideal of R.
2. For every finite family $\alpha_i \leq i \leq n$ of elements of A, there exists $t \in A$ such that $\alpha_i = \alpha_i t\ \forall\ i, 1 \leq i \leq n$.
3. For all $\alpha \in A$ there exists $\beta \in A$ such that $\alpha = \alpha\beta$.
4. $\frac{R}{A1 + A2}$, be a U-flat R-module".

Moreover, if A is finitely generated, then A is pure if and only if it is generated by an idempotent[4].

Proof: (2) $\Rightarrow$ (3), is obvious.

(3)$\Rightarrow$(4), let B be an ideal of R, we must prove that $A \cap B = A.B$ if $a \in (A_1 + A_2) \cap B$ there exists $t \in (A_1 + A_2)$ such that $\alpha = \alpha t$. Hence $\alpha \in (A_1 + A_2)$ B[23].

(4) $\Rightarrow$ (3), if $\alpha \in A_1 + A_2$, then $R\alpha = (A_1 + A_2) \cap (A_1 + A_2)\alpha = A\alpha$[23].

(1) $\Rightarrow$ (3), if $\alpha \in A_1 + A_2$, 1 is solution of the equation $\alpha x = \alpha$. So this equation has a solution in $A_1 + A_2 = A$[23].

(3) $\Rightarrow$ (2) Let $\alpha_1, \alpha_2, \alpha_2, \ldots \alpha_n$ be the element of $(A_1 + A_2) = A$, we proceed by induction on n. There exist

t ∈ $(A_1 + A_2)$ such that α_m = t α_m. By induction hypothesis there exist S ∈ $A_1 + A_1$ such that, αn - t α_n = s(α_n - t α_n) ∀ *i* ; 1≤ *i* ≤ (n-1). Now, it is easy to check that (s + t - st) $\alpha_{i} = \alpha_i$ ∀ *i* ; 1≤ *i* ≤ n[23].
(2) ⇒ (1), we consider the subsequent system of equations, $\sum_{i-1}^{n} r_{j.i}\, r_i = \alpha_j \in A, I \leq J \leq P$. Assume that ($\beta_1$, β_2 β_3,...... β_n) be a solution of this system in R, then there exists S ∈ $(A_1 + A_2)$ such that α_j = S α_j ∀ *i* ; 1≤ *i* ≤ P, so (sα_1, sα_2, sα_3,............... sα_n) be a solution of this system in $A_1 + A_2 = A$[23].

Lemma 1.5: Let 0→A→B→C→ 0 be an exact sequence such that A and C are strongly U-Flat modules, then B is strongly U-flat modules[13].

Proof: Let M be a strongly U-flat module, by induced exact sequence, Ext_R^1 (C ⊗ $(M_1 + M_2)$) → Ext_R^1 (B ⊗ $(M_1 + M_2)$) → Ext_R^1 (A ⊗ $(M_1 + M_2)$) Since Ext_R^1 (C ⊗ $(M_1 + M_2)$) = 0 and Ext_R^1(A ⊗$(M_1 + M_2)$) = 0 then Ext_R^1(B ⊗ $(M_1 + M_2)$) = 0[13].

Example 1.6: The Z- module Q is strongly flat modules recalls that R is called a Matlis domain if the projective Dimension of Q (or equivalently k) is one, module C is called Matlis cotorsion if Ext_R^1 (Q ⊗ C) =0 and M is called strongly U- flat if Ext_R^1(($M_1 + M_2$) ⊗ C) = 0 for every Matlis cotorision R-module C[6].

Corollary 1.7: Let R be a semi-Dedekind domain. If M be a U-projective R-module and N is weak U-projective R-module then $(M_1 + M_2)$ $\otimes_R^N$, be a weak W-projective[12].

Proof: The isomorphism Tor_n^R((M_1 +M_2) ⊗ N, A) ≅ (M_1 +M_2) ⊗ Tor_n^R(N, A) together with pure U-projective and pure U-injective σ($M_1 + M_2$) : ($M_1 + M_2$) → E($M_1 + M_2$) denotes the pure injective envelope of an R-module M where M = $M_1 + M_2$. Recall that an injective envelope σ($M_1 + M_2$) : ($M_1 + M_2$) → E($M_1 + M_2$) has the unique mapping property, if for an homomorphism f : ($M_1 + M_2$) → N with ker f = 0 and N pure injective. Then there exist a unique homomorphism g: E($M_1 + M_2$) →E such that g σ m : f. where $M_1 + M_2 = M$[12].

Theorem 1.8: Subsequent statements are equivalent[2];
1. "R be a Prufer domain
2. Every R-module is pure U- injective.
3. Ext_R^1($M_1 + M_2$, N) = 0 for all pure U-injective R-module N.
4. Every pure U-injective R-module has an injective envelope with unique mapping property"[2].

Proof: (a) ⇒ (b), it is easy to verify[2].
(b) ⇒ (c), if every R-module is pure w-projective then Ext_R^1 ($M_1 + M_2$,N) = 0[2].
(d) ⇒ (a), let M be a pure U-injective R-module. We have the subsequent exact commutative diagram;

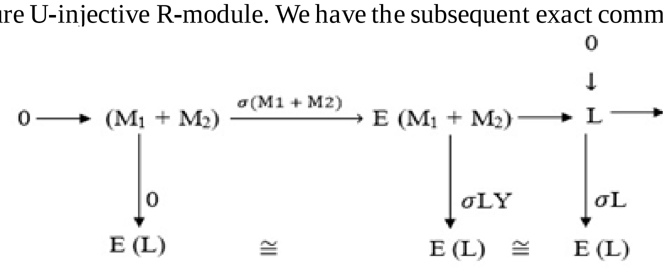

Figure 1.2: Diagram to explain exact commutative property[2].

Note that $\sigma L\gamma M = 0 = \sigma M$, So $\sigma L\gamma M = 0$ in view of the (d), therefore $L = im(\gamma) \subseteq ker(\sigma L) = 0$ and hence M is pure U-injective where $M = M_1 + M_2$. Let I be a class of R-module and M be an R-module. A homomorphism $\phi \in Hom_R(N, M)$ with $N \in I$ is called an I pure pre-cover of M, if the induces map $Hom_R(I_N, \phi) : Hom_R(N^1, N) \to Hom_R(N^1, M_1 + M_2)$ is surjective for all $N^1 \in l$. An I- pure cover $\phi \in Hom_R(N, M + M_2)$ is called an I- pure cover if every $\gamma \in Hom_R(N, M_1 + M_2)$ is called an I- pure pre-cover. If I is the class of pure U- injective R-module, then an I- envelope is called a pure U- injective envelope[2].

Proposition 1.9: If M be an R-module, then the subsequent statements are equivalent[6];

1. "M is pure U- projective; where $M = M_1 + M_2$
2. M is pure projective concerning every exact sequence $0 \to A \to B \to C \to 0$, where A is pure U- projective.
3. For every exact sequence, $0 \to K \to F \to M \to 0$ with ker t = 0 where F is pure U-injective, $K \to F$ be a pure U-injective pre-envelope of K.
4. M is co-kernel of a pure U-injective pre-envelope of $K \to F$ with F projective"[6].

Proof: (1) $\Rightarrow$ (2), let $0 \to A \to B \to C \to 0$ be an exact sequence where A is pure U- injective. Then $Ext_R^1(M_1 + M_2 + A) = 0$ by (5), Therefore $Hom_R(M_1 + M_2, B) \to Hom_R(M_1 + M_2, C) \to 0$ is exact, and (2) holds[6].

(2) $\Rightarrow$ (1), for every pure U-injective R-module N, there be a short exact sequence $0 \to N \to E \to L \to 0$ with E injective which induces an exact sequence, $Hom_R(M_1+M_2+E) \to Hom_R(M_1+M_2+L) \to Ext_R^1(M_1+M_2+N) \to 0$. Since, $Hom_R(M_1+M_2+E) \to Hom_R(M_1+M_2+L) \to 0$ is exact by (6), we have $Ext_R^1(M_1 + M_2 + N) = 0$ and (5) follows[6].

(1) $\Rightarrow$ (3), it is easy to verify[6].

(3) $\Rightarrow$ (4), let $0 \to K \to P \to M \to 0$ be an exact sequence with P-pure projective and $M = M_1 + M_2$. Here P is pure W- injective by hypothesis; thus, $K \to P$ be a pure U-injective pre-envelope[6].

(4) $\Rightarrow$ (1), there be an exact sequence $0 \to K \to P \to M \to 0$ where $K \to P$ be a pure U-injective pre-envelope with P pure projective. It gives rise to the exactness of $Hom_R(P, N) \to Hom_R(K, N) \to Ext_R^1(M, N) \to 0$ for every pure U- injective R-module N. Note that $Hom_R(P, N) \to Hom_R(K, N) \to 0$ is exact by (8). Hence $Ext_R^1(M, N) \to 0$, as desired , where $M = M_1 + M_2$ [6].

2. Pure U-projective Dimension over the semi-Dedekind Domain :

Definition: (1) "For any R-module M, let pure U-injective dimension P wid(M) of M, denote the smallest integer $n \geq 0$ such that $Ext_R^{n+1}(N,M) = 0$ for every R-module N of weak dimension ≤ 1. (If no such n exists, set P wid(M) = ∞)"[17].

(2) "P wid(R) = Sup {P wid(M) : M be an R-module}"[17].

Lemma 2.1: Let R be a semi-Dedekind domain then for an R-module M, the subsequent statements are equivalent[17];

1. "P wid (M) $\leq$ n.
2. $Ext_R^{n+1}(N, M) = 0$ for all R-module N of U- dimension ≤ 1.
3. If the sequence $0 \to M \to E_0 \to E_1 \to \ldots\ldots\ldots\ldots \to E_n \to O$ is exact with $E_0, E_1, \ldots\ldots E_{n-1}$ pure U-injective, then also E_n is pure U- injective".

Proof: (1) $\Rightarrow$ (2), using induction on n, it is clear that if P wid(M) $\leq$ n-1 resolve N by $0 \to K \to P \to N \to 0$ with K and P flat, K have P w-dimension ≤ 1 and Ext_R^{n+1} (N, M) $\cong$ Ext_R^1 (K , M) = 0 by induction hypothesis[17].
(2) $\Rightarrow$ (3), follows from the isomorphism Ext_R^{n+1} (N, M) $\cong$ Ext_R^1 (N, E_n)[17].
(2) $\Rightarrow$ (1), trivial[17].

Proposition 2.2: Let R be a semi-Dedekind domain. For R-module M and an integer n≥0, the subsequent are equivalent[6,14]:

1. "P wid(M) $\geq$ n.
2. Ext_R^{n+1}(N, M) = 0 for any pure U-injective R-module N.
3. Ext_R^{n+1}(M, N) = 0 for any pure U-injective R-module N. There exists an exact sequence, $0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to O$ where every P_i is pure U-projective".

Proof: (2) $\Rightarrow$ (3**)**, for any pure U-injective R-module, N, there be a short exact sequence $0 \to N \to E \to L \to 0$, where E is injective. Then the sequence Ext_R^{n+1} (N, L) $\to$ Ext_R^{n+1}(M, E) = 0 is exact. Note that L is pure U-injective so Ext_R^{n+1} (M, N) = 0 by (2) hence Ext_R^{n+1}(M, N) = 0[6,14].

(1) $\Rightarrow$ (2), is similar to (1) $\Rightarrow$ (3).
(1) $\Leftrightarrow$ (4), is straightforward.
(2) $\Rightarrow$ (1), is obvious.

3. Conclusion

Hence these results will generalize the concept of Strongly U-Flat Modules over Matlis Domains. In this study, we found that a ring R with a local is U-flat as a left R-module and finitely generated, and then it is pure if and only if generated by an idempotent. In $O \to A \to B \to C \to O$, exact sequences A, C are strongly U-Flat modules, and then B is strongly U-flat modules. Relations and various definitions have discussed, which explains more about R-module.

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