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A study of Fibonacci & Lucas Vectors

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Abstract

An attempt has been made to put forth certain properties of Lucas and Fibonacci vectors and establish a relationship between the vectors using a special matrix. Cross products between Fibonacci and Lucas vectors have been investigated.

Also, it was observed that, there exists a homeomorphism between the Fibonacci plane and any plane parallel to it.

Key words : Fibonacci and Lucas numbers, position vectors, vector product, scalar triple product.
Mathematics Subject Classification (2000) : 11B39

1 Introduction

1.1 Definition

A vector is a quantity having both magnitude and direction. It is denoted by a directed line segment. The length of the segment denotes the magnitude of the vector and the direction is shown by the unit vectors acting along the x,y and z axes, namely $\hat{i}$, $\hat{j}$ and $\hat{k}$.

1.2 Fibonacci and Lucas Vectors :

Here, we shall discuss some special vectors in space which are in the form $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$, where x, y and z are the direction ratios and are denoted by consecutive Fibonacci or Lucas numbers.

Consider three consecutive Fibonacci numbers F_n, F_{n+1}, F_{n+2} or three Lucas numbers L_n, L_{n+1}, L_{n+2} denoted by x , y and z respectively.

Since a Fibonacci number is obtained by adding the two previous Fibonacci numbers or a

Lucas number is obtained by adding two previous Lucas numbers, we have

$F_{n+2} = F_n + F_{n+1}$ or $L_{n+2} = L_n + L_{n+1}$ (1)

hence we get

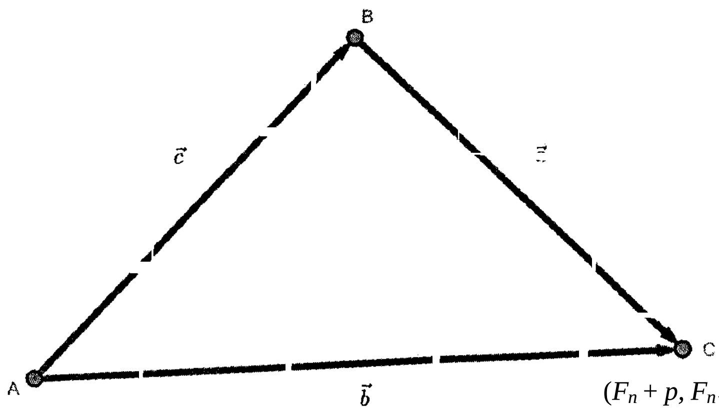
$$z = x + y \qquad (2)$$

$\Rightarrow$ $x + y - z = 0$ represents a plane through the origin containing all Fibonacci position vectors (F_n, F_{n+1}, F_{n+2}) or (F_{n+1}, F_n, F_{n+2}) or all Lucas vectors (L_n, L_{n+1}, L_{n+2}) or (L_{n+1}, L_n, L_{n+2})

2 Scalar Triple Product :

The fact that the points (F_n, F_{n+1}, F_{n+2}) , (L_n, L_{n+1}, L_{n+2}) , (F_{n+1}, F_n, F_{n+2}) and (L_{n+1}, L_n, L_{n+2}) lie in the plane is reiterated by proving a scalar triple product or box product to be zero , as shown below.

Let the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ be denoted by



$\vec{a} = (F_{n+k} - F_n)\,\hat{i} + (F_{n+k+1}) - F_{n+1})\,\hat{j} + (F_{n+k+2} - F_{n+2})\,\hat{k}$

$\vec{b} = (F_{n+p} - F_{n+k})\,\hat{i} + (F_{n+p+1} - F_{n+k+1})\,\hat{j} + (F_{n+p+2} - F_{n+k+2})\,\hat{k}$

$\vec{c} = (F_{n+p} - F_n)\,\hat{i} + (F_{n+p+1} - F_{n+1})\,\hat{j} + (F_{n+p+2} - F_{n+2})\,\hat{k}$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} F_{n+k} - F_n & F_{n+k+1} - F_{n+1} & F_{n+k+2} - F_{n+2} \\ F_{n+p} - F_{n+k} & F_{n+p+1} - F_{n+k+1} & F_{n+p+2} - F_{n+k+2} \\ F_{n+p} - F_n & F_{n+p+1} - F_{n+1} & F_{n+p+2} - F_{n+2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & F_{n+k+1} - F_{n+1} & F_{n+k+2} - F_{n+2} \\ 0 & F_{n+p+1} - F_{n+k+1} & F_{n+p+2} - F_{n+k+2} \\ 0 & F_{n+p+1} - F_{n+1} & F_{n+p+2} - F_{n+2} \end{vmatrix} C_1 \to C_1 + C_2 - C_3$$

$\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are coplanar.

We know that in a 2D plane, a Fibonacci vector is denoted by $[F_{n+1} \quad F_n]$ and a Lucas vector by $[L_{n+1} \quad L_n]$. Now, using an R matrix, that is $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ we can transform a Lucas vector into a Fibonacci vector.

$$[L_{n+1} \; L_n].\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = [L_{n+1} + 2L_n \quad 2L_{n+1} - L_n]$$

$$= [5F_{n+1} \quad 5F_n]$$

$$= 5\,[F_{n+1} \quad F_n]$$

Theorem 1 :

In a 3D vector space a Fibonacci vector $[F_{n+1} \quad F_n \quad F_{n+2}]$ is transformed into a Lucas vector $[L_{n+1} \quad L_{n+2} \quad L_{n+3}]$ when multiplied by the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Proof

$$[F_{n+1} \quad F_n \quad F_{n+2}]\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} = [F_{n+1} + 2F_n \quad 2F_{n+1} + F_{n+2} \quad F_{n+1} + 3F_{n+2}]$$

$$= [L_{n+1} \; L_{n+2} \; L_{n+3}]$$

Theorem 2 :

A Fibonacci vector $[F_n \quad F_{n+1} \quad F_{n+2}]$ is transformed into a Lucas vector $[L_{n+1} \quad L_{n+2} \quad L_{n+3}]$ when multiplied by the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Proof :

$$[F_n \quad F_{n+1} \quad F_{n+2}].\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} = [2F_n + F_{n+1} \quad 2F_{n+1} + F_{n+2} \quad F_{n+1} + 3F_{n+2}]$$

$$= [L_{n+1} \quad L_{n+2} \quad L_{n+3}]$$

also,

$$|L| = 3.6 \times |F| \tag{3}$$

where $|F| = \sqrt{F_{n+1}^2 + F_n^2 + F_{n+2}^2}$ and $|L| = \sqrt{L_{n+1}^2 + L_{n+2}^2 + L_{n+3}^2}$

If $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$, *then* $A^{-1} = \frac{1}{10}\begin{bmatrix} 5 & 0 & 0 \\ -3 & 6 & -2 \\ 1 & -2 & 4 \end{bmatrix}$

$$F.A = L$$
$$\Rightarrow L.A^{-1} = F$$

$$[L_{n+1}\ L_{n+2}\ L_{n+3}] . \frac{1}{10}\begin{bmatrix} 5 & 0 & 0 \\ -3 & 6 & -2 \\ 1 & -2 & 4 \end{bmatrix} = \frac{1}{10}[5L_{n+1} - 3L_{n+2} + L_{n+3} \quad 6L_{n+2} - 2L_{n+3} \quad -2L_{n+2} + 4L_{n+3}]$$

$$= [F_n \quad F_{n+1} \quad F_{n+2}]$$

3 Homeomorphism :

The direction ratios of the normal to the plane $x + y - z = 0$ are 1, 1, −1.

The equation to the normal through any arbitrary point $(F_n, \quad F_{n+1}, \quad F_{n+2})$ is

$$\frac{x - F_n}{1} = \frac{y - F_{n+1}}{1} = \frac{z - F_{n+2}}{-1} = k$$

$$x = Fn + k, \quad y = F_{n+1} + k, \quad z = F_{n+2} - k \tag{4}$$

Now, let there be a plane parallel to the Fibonacci plane $x + y - z = 0$ as $x + y - z = \mu$ The normal line intersects the plane $x + y - z = \mu$ at $(F_n + k, \quad F_{n+1} + k, \quad F_{n+2} - k)$

$$\text{Hence } x + y - z = \mu$$
$$\Rightarrow (F_n + k) + (F_{n+1} + k) - (F_{n+2} - k) = \mu$$
$$\Rightarrow 3k = \mu$$
$$\Rightarrow k = \frac{\mu}{3}$$

Any point $(F_n, \ F_{n+1}, \ F_{n+2})$ on $x+y-z = 0$ have an image $(F_n + \frac{\mu}{3}, \ F_{n+1} + \frac{\mu}{3}, \ F_{n+2} - \frac{\mu}{3})$ on the plane $x + y - z = \mu$ (5)

Similarly, any point $(L_n, \quad L_{n+1}, \quad L_{n+2})$ on $x+y-z = 0$ have an image $(L_n + \frac{\mu}{3}, \quad L_{n+1} + \frac{\mu}{3}, \quad L_{n+2} - \frac{\mu}{3})$ on the plane $x + y - z = \mu$ (6)

Hence, the plane $x + y - z = \mu$ is homeomorphic to the Fibonacci plane $x + y - z = 0$

4. *Vector Product :*

Consider the vectors ,

$$\vec{F}_1 = F_n\hat{i} + F_{n+1}\hat{j} + F_{n+2}\hat{k}, \qquad \vec{F}_2 = F_{n+1}\hat{i} + F_n\hat{j} + F_{n+2}\hat{k}$$
$$\vec{L}_1 = L_n\hat{i} + L_{n+1}\hat{j} + L_{n+2}\hat{k}, \qquad \vec{L}_2 = L_{n+1}\hat{i} + L_n\hat{j} + L_{n+2}\hat{k}$$

Theorem 3

$$\vec{F}_1 \times \vec{L}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_n & F_{n+1} & F_{n+2} \\ L_n & L_{n+1} & L_{n+2} \end{vmatrix}$$

$$= [F_{n+1}L_{n+2} - L_{n+1}F_{n+2}]\,\hat{i} + [F_{n+2}L_n - L_{n+2}F_n]\,\hat{j} + [F_nL_{n+1} - L_nF_{n+1}]\,\hat{k}$$

$$= [F_{n+1}L_n - L_{n+1}F_n]\,\hat{i} + [F_{n+1}L_n - L_{n+1}F_n]\,\hat{j} + [F_nL_{n+1} - L_nF_{n+1}]\,\hat{k}$$

$$= [F_{n+1}L_n - L_{n+1}F_n][\,\hat{i} + \hat{j} - \hat{k}\,]$$

$$= [\,\frac{\{\alpha^{n+1} - \beta^{n+1}\}}{\{\alpha - \beta\}}\,.\,\{\alpha^n + \beta^n\} - \{\alpha^{n+1} + \beta^{n+1}\}\,\frac{\{\alpha^n - \beta^n\}}{\{\alpha - \beta\}}\,][\,\hat{i} + \hat{j} - \hat{k}\,]$$

$$= (-1)^n.2.[\,\hat{i} + \hat{j} - \hat{k}\,]$$

Theorem 4 :

$$\vec{F}_2 \times \vec{L}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_{n+1} & F_n & F_{n+2} \\ L_{n+1} & L_n & L_{n+2} \end{vmatrix}$$

$$= [F_nL_{n+2} - L_nF_{n+2}]\,\hat{i} + [F_{n+2}L_{n+1} - L_{n+2}F_{n+1}]\,\hat{j} + [F_{n+1}L_n - L_{n+1}F_n]\,\hat{k}$$

$$= [F_nL_{n+1} - L_nF_{n+1}][\,\hat{i} + \hat{j} - \hat{k}\,]$$

$$= [\,\frac{\{\alpha^n - \beta^n\}}{\{\alpha - \beta\}}\,.\{\alpha^{n+1} + \beta^{n+2}\} - \{\alpha^n + \beta^n\}\,\frac{\{\alpha^{n+1} - \beta^{n+1}\}}{\{\alpha - \beta\}}\,][\,\hat{i} + \hat{j} - \hat{k}\,]$$

$$= (-1)^{n+1}.2.[\,\hat{i} + \hat{j} - \hat{k}\,]$$

Theorem 5 :

$$\vec{F}_1 \times \vec{L}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_n & F_{n+1} & F_{n+2} \\ L_{n+1} & L_n & L_{n+2} \end{vmatrix}$$

$$= [F_{n+1}L_{n+2} - L_nF_{n+2}]\,\hat{i} + [F_{n+2}L_{n+1} - L_{n+2}F_n]\,\hat{j} + [F_nL_n - L_{n+1}F_{n+1}]\,\hat{k}$$

$$= [F_{n+1}L_{n+1} - L_nF_n][\,\hat{i} + \hat{j} - \hat{k}\,]$$

$$= F_{2n+1}[\,\hat{i} + \hat{j} - \hat{k}\,]$$

Theorem 6 :

$$\vec{F}_2 \times \vec{L}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_{n+1} & F_n & F_{n+2} \\ L_n & L_{n+1} & L_{n+2} \end{vmatrix}$$

$$= [F_nL_{n+2} - L_{n+1}F_{n+2}]\,\hat{i} + [F_{n+2}L_n - L_{n+2}F_{n+1}]\,\hat{j} + [F_{n+1}L_{n+1} - L_nF_n]\,\hat{k}$$

$$= [F_nL_n - L_{n+1}F_{n+1}][\hat{i} + \hat{j} - \hat{k}]$$

$$= (-1)F_{2n+1}[\hat{i} + \hat{j} \text{ “ } \hat{k}]$$

Conclusions

In this work, an investigation was done on Fibonacci and Lucas vectors. Two transformation matrices were created to transform Fibonacci vectors to Lucas vectors. A homeomorphism was established between planes parallel to the Fibonacci plane. Vector products between Fi- bonacci and Lucas vectors were investigated and four results were obtained in the process.

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