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A note on fuzzy continuous mappings in fuzzy biclosure spaces

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Abstract

Here in this paper we study various types of continuity in fuzzy biclosure spaces. In particular we introduce fuzzy continuous map, fuzzy bicontinuous map and fuzzy strong pairwise continuous map in fuzzy biclosure spaces. We compare them with each other and we also obtain some relevant results.

Key words : Fuzzy closure operator, fuzzy biclosure space, fuzzy continuous map, fuzzy bicontinuous map, fuzzy strong pairwise continuous mapping.

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1- Introduction

The notions of closure spaces were introduced by Birkhoff [1] and Cech[6] independently. Later on Boonpok[2] introduced the notion of biclosure spaces. Such spaces are equipped with two arbitrary closure operators. Further the concept of fuzzy closure spaces has been introduced by Mashhour and Ghanim[8] and Srivastava *et al.*[11]. Mashhour and Ghanim generalize the concept of Cech closure spaces while Srivastava *et al* generalizes the concept of Birkhoff closure spaces. Later on Tapi and Navalakhe[14] introduced and studied the concept of fuzzy biclosure spaces in detail.

In this paper we have introduced the concept of fuzzy biclosure space as generalization of Srivastava *et al.*[9] and introduced and studied different types of continuity viz. fuzzy continuous mapping, fuzzy bicontinuous mapping and fuzzy strong pairwise continuous mapping in fuzzy biclosure spaces.

2- Preliminaries :

Definition 2.1[9] *:* A function $c: I^x \to I^x$ is called a fuzzy closure operation on X if it satisfies the following conditions:

i. $c(\underline{\alpha}) = \underline{\alpha}, \alpha \in [0,1]$

ii. $A \subseteq c(A), \forall A \in I^x$

iii. $A \subseteq B \Rightarrow c(A) \subseteq c(B), \quad \forall A, B \in I^x$

iv. $c(c(A)) = c(A), \forall A, B \in I^x$

The pair (X, c) is called a fuzzy closure space. It is called fuzzy closure operation on X. This definition is obviously an analogue of Birkhoff closure operator.

A closure operator c on a set X is called additive (respectively, idempotent) if $A, B \subseteq X \Rightarrow c(A \cup B) = c(A) \cup c(B)$ (respectively, $A \subseteq X \Rightarrow c(c(A)) = c(A)$). A subset $A \subseteq X$ is closed in the closure space (X,c) if $c(A) = A$ and it is open if its complement in X is closed. The empty set and the whole space are both open and closed. A closure space $(Y,c*)$ is said to be a subspace of (X, c) if $Y \subseteq X$ and $c^*(A) = c(A) \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, c). Then the subspace $(Y,c*)$ of (X, c) is said to be closed too. Let (X,c) and $(Y,c*)$ be closure spaces. A map $f: (X, c) \rightarrow (Y, c^*)$ is said to be continuous if $f(c(A)) \subseteq c^* f(A)$ for every subset $A \subseteq X$.

A map $f: (X, c) \rightarrow (Y, c^*)$ is continuous iff $cf^{-1}(B) \subseteq f^{-1}(c^* B)$ for every subset $B \subseteq Y$.

Clearly, if $f: (X, c) \rightarrow (Y, c^*)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, c) for every closed subset F of $(Y,c*)$.

Let (X, c) and $(Y,c*)$ be closure spaces. A map $f: (X, c) \rightarrow (Y, c^*)$ is said to be closed (respectively open) if $f(F)$ is closed (respectively open) subset of $(Y,c*)$ whenever F is a closed (respectively open) subset of (X, c).

3- *Fuzzy Biclosure spaces:*

Definition 3.1[12]*:* A function $c_i: I^x \rightarrow I^x$, $i = 1, 2$ is called a fuzzy biclosure operation on X if the following axioms are satisfied:

i. $c_i(\underline{\alpha}) = \underline{\alpha}, \alpha \in [0,1]$

ii. $A \subseteq c_i(A), \forall A \in I^x$

iii. $A \subseteq B \Rightarrow c_i(A) \subseteq c_i(B), \quad \forall A, B \in I^x$

iv. $c_i(c_i(A)) = c_i(A), \forall A, B \in I^x$

Definition 3.2: A subset A of a fuzzy biclosure space (X, c_1, c_2) is said to be fuzzy closed if:

$$c_1(c_2(A)) = A$$

The complement of fuzzy closed set is known as fuzzy open set.

Here we follow the definition of fuzzy biclosure space and other relevant definition from Srivastava[11], Viriyapong *et.al.*[15]

Definition 3.3: A fuzzy biclosure space is a triple (X, c_1, c_2) where X is a set and c_1, c_2 are two arbitrary fuzzy closure operators on X.

Definition 3.4: A subset A of a fuzzy biclosure space (X, c_1, c_2) is called fuzzy closed if $c_1(c_2(A)) = A$. The complement of fuzzy closed set is called fuzzy open.

Clearly, A is a fuzzy closed subset of a fuzzy biclosure space (X, c_1, c_2) iff A is fuzzy closed subset of (X, c_1) *and* (X, c_2).

Let A be a closed subset of fuzzy biclosure space (X, c_1, c_2). The following conditions are equivalent.

(i) $c_1(c_2(A)) = A$

(ii) $c_1(A) = A, \; c_2(A) = A$.

Definition 3.5[11]*:* A fuzzy biclosure space (X, c_1, c_2) is called Hausdorff if $\forall$ pair of distinct fuzzy points x_r, y_s in X, $\exists\; U, V \in c_1, c_2$ such that $x_r \in U, y_s \in V$ and $U \cap V = \phi$.

Definition 3.6[15] *:* Let (X, c_1, c_2) and (Y, c_1^*, c_2^*) be two fuzzy biclosure spaces and $f: (X, c_1, c_2) \to (Y, c_1^*, c_2^*)$ be a map. Then f is said to be fuzzy continuous (in short, F-continuous) if the inverse image of each c_i^*- fuzzy open set (closed set) is c_i-fuzzy open set (closed set) for $i = 1,2$.

Definition 3.7[15]*:* Let (X, c_1, c_2) and (Y, c_1^*, c_2^*) be two fuzzy biclosure spaces and $f: (X, c_1, c_2) \to (Y, c_1^*, c_2^*)$ be a map. Then f is said to be fuzzy pairwise continuous (in short, fuzzy P-continuous) if the maps $f_i: (X, c_i) \to (Y, c_i^*)$ are fuzzy continuous for $i = 1,2$.

Proposition 3.1: Let (X, c_1, c_2) be a fuzzy biclosure space. If A *and* B are fuzzy closed subset of (X, c_1, c_2). Then $A \cap B$ is also fuzzy closed.

Proof : The proof is obvious.

Proposition 3.2: Let (X, c_1, c_2) be a fuzzy biclosure space and c_1 *and* c_2 be additive. If A *and* B are fuzzy closed subset of (X, c_1, c_2). Then $A \cup B$ is also fuzzy closed.

Proof : Since A *and* B are fuzzy closed subset of (X, c_1, c_2) since c_1 *and* c_2 are said to satisfy additive property. Then $c_1(c_2(A)) = A$ and $c_1(c_2(B)) = B$

Now consider, $c_1(c_2(A \cup B)) = c_1(c_2(A) \cup c_2(B))$

$= c_1(c_2(A)) \cup c_1(c_2(B))$

$= A \cup B$

Therefore, $A \cup B$ is fuzzy closed if c_1 *and* c_2 are additive.

Definition 3.8: Let (X, c_1, c_2) be a fuzzy biclosure space. A fuzzy biclosure space (Y, c_1^*, c_2^*) is called a subspace of (X, c_1, c_2) if $Y \subseteq X$ and $c_i^* A = c_i A \cap Y$ for each $i \in \{1, 2\}$ and each subset $A \subseteq Y$.

Proposition 3.3: Let (X, c_1, c_2) be a fuzzy biclosure space and let (Y, c_1^*, c_2^*) be fuzzy closed subspace of (X, c_1, c_2). If F is a fuzzy closed subset of (Y, c_1^*, c_2^*). The F is a fuzzy closed subset of (X, c_1, c_2).

Proof: Let F be a fuzzy closed subset of (Y, c_1^*, c_2^*). Then $c_1^*(F) = F$ and $c_2^*(F) = F$. Since Y is a closed fuzzy subset of (X, c_1) and (X, c_2) both $c_1(F) = F$ and $c_2(F) = F$. Consequently, F is a fuzzy closed subset of (X, c_1) and (X, c_2) both. Therefore, F is a fuzzy closed subset of (X, c_1, c_2).

4- Fuzzy continuous mappings :

Definition-4.1: Let (X, c_1, c_2) and (Y, c_1^*, c_2^*) be fuzzy biclosure spaces. A mapping $f\colon (X, c_1, c_2) \to (Y, c_1^*, c_2^*)$ is said to be fuzzy continuous if $f(c_i A) \subseteq c_i^* f(A),\ \forall\ A \subseteq X, i = \{1, 2\}$.

Theorem 4.1: Let (X, c_1, c_2) be a fuzzy biclosure spaces A and let (Y, c_1^*, c_2^*) be a Hausdorff space. Let $f\colon X \to Y$ be a one-one continuous map. Then (X, c_1, c_2) is also Hausdorff space.

Proof : Let x_1, x_2 be any two distinct points of (X, c_1, c_2). Since f is one-one, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. Let $x_r = f(x_1), y_s = f(x_2)$. So that $x_r \neq y_s$. Since (Y, c_1^*, c_2^*) is Hausdorff then there exist pair of distinct fuzzy points x_r and y_s in Y. There exist $U, V \in c_1^*, c_2^*$ such that $x_r \in U, y_s \in V$ and $U \cap V = \phi$.

Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are T- open. Now, $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\phi) = \phi$, and $x_r \in U \Rightarrow f^{-1}(x_r) \in f^{-1}(U) \Rightarrow x_r \in f^{-1}(U)$ and $y_s \in V \Rightarrow f^{-1}(y_s) \in f^{-1}(V) \Rightarrow y_s \in f^{-1}(V)$. Thus it is shown that for every pair of distinct points x_1, x_2 of X, there exist disjoint T-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that, $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$ accordingly, the space (X, c_1, c_2) is also hausdorff space.

Theorem 4.2: A mapping f of a fuzzy biclosure space (X, c_1, c_2) on to another space (Y, c_1^*, c_2^*) is continuous iff $c_i(f^{-1}(B)) \subset f^{-1}(c_i^*(B))$ for every $B \subset Y$.

Proof : Let f be continuous and let $B \subset Y$. Since $c_i^*(B)$ is closed in (Y, c_1^*, c_2^*) and $f^{-1}(c_i^*(B))$ is closed in (X, c_1, c_2). and therefore

$$c_i(f^{-1}(B)) = f^{-1}(c_i^*(B)) \tag{i}$$

Now, $B \subset c_i^*(B) \Rightarrow f^{-1}(B) \subset f^{-1}(c_i^*(B))$ $\{\because f^{-1}(B)(x) \le f^{-1}(c_i^*(B))(x)\}$

$c_i(f^{-1}(B))(x) \le c_i(f^{-1}(c_i^*(B)))(x) = f^{-1}(c_i^*(B))(x)$ {By using (i)}

$$c_i(f^{-1}(B)) \subset c_i(f^{-1}(c_i^*(B)) \tag{ii}$$

Conversely- Let the condition hold and let F be any closed set in (Y, c_1^*, c_2^*) so that $c_i^* F = F$. By hypothesis $c_i(f^{-1}(F)) \subset f^{-1}(c_i^*(F)) = f^{-1}(F)$.

But, $f^{-1}(F)(x) \le c_i^*(f^{-1}(F))(x) \quad \forall\, x \in X,$ always

Hence, $c_i(f^{-1}(F)) = f^{-1}(F)$ and so $f^{-1}(F)$ is closed in (X, c_1, c_2). It follows that f is continuous.

Proposition 4.1: Let (X, c_1, c_2) and (Y, c_1^*, c_2^*) be fuzzy biclosure spaces. If mapping $f: (X, c_1, c_2) \to (Y, c_1^*, c_2^*)$ is fuzzy continuous, then $f^{-1}(G)$ is an fuzzy open subset of (X, c_1, c_2) $\forall$ open subset G of (Y, c_1^*, c_2^*).

Proof : This proof is obvious.

Proposition 4.2: Let $(X, c_1, c_2), (Y, c_1^*, c_2^*)$ and (Z, c_1^{**}, c_2^{**}) are fuzzy biclosure spaces. If mapping $f: (X, c_1, c_2) \to (Y, c_1^*, c_2^*)$ and $g: (Y, c_1^*, c_2^*) \to (Z, c_1^{**}, c_2^{**})$ are fuzzy continuous, then $gof: (X, c_1, c_2) \to (Z, c_1^{**}, c_2^{**})$ is fuzzy continuous.

Proof : Let $A \subseteq X$. Since $gof(c_iA) = g(f(c_iA))$ and f fuzzy continuous, $(f(c_iA)) \subseteq (c_i^* f(A))$. Again Since *g* is fuzzy continuous then we get -

$g(f(c_iA)) \subseteq g(c_i^* f(A))$ { $f(A)$ as a subset of Y} (1)

$g(c_i^* f(A)) \subseteq c_i^{**} g(f(A)) \ldots\ldots\ldots(2)$,

Combining both (1) and (2) we get,

$$g(f(c_iA)) \subseteq c_i^{**} g(f(A))$$

Consequently, gof is fuzzy continuous.

Proposition 4.3: Let (X, c_1, c_2) and (Y, c_1^*, c_2^*) be fuzzy biclosure spaces and let (A, c_{1A}, c_{2A}) be a fuzzy closed subspace of (X, c_1, c_2). If $f: (X, c_1, c_2) \to (Y, c_1^*, c_2^*)$ is fuzzy continuous, then $f/A: (A, c_{1A}, c_{2A}) \to (Y, c_1^*, c_2^*)$ is fuzzy continuous.

Proof : If $B \subseteq A$, then

$$f/A\ (A, c_{1A}, c_{2A}) = (f/A)(c_iB \cap A) = (f/A)\ (c_iB)$$
$$= f\ (c_iB) \subseteq c_i^*\ f\ (B) = c_i^*\ (f/A\ (B))$$
$$\Rightarrow f/A\ (A, c_{1A}, c_{2A}) \subseteq c_i^*\ (f/A)\ (B)$$

Hence, f/A is fuzzy continuous.

Definition 4.2: Let (X, c_1, c_2) and (Y, c_1^*, c_2^*) be fuzzy biclosure spaces. A mapping $f: (X, c_1, c_2) \to (Y, c_1^*, c_2^*)$ is said to be fuzzy closed (res fuzzy open) if f (F) is fuzzy closed (res. fuzzy Open) subset of (Y, c_1^*, c_2^*) whenever F is a fuzzy closed (res. fuzzy Open) subset of (X, c_1, c_2).

5- Fuzzy Bicontinuous Mappings :

Definition 5.1: Let (X, c_1, c_2) and (Y, c_1^*, c_2^*) be fuzzy biclosure spaces. A map $f: (X, c_1, c_2) \to (Y, c_1^*, c_2^*)$ is called fuzzy bicontinuous if the map $f: (X, c_1) \to (Y, c_2^*)$ and $f: (X, c_2) \to (Y, c_1^*)$ are fuzzy continuous.

Proposition 5.1: Let (X, c_1, c_2) and (Y, c_1^*, c_2^*) be fuzzy biclosure spaces. Then $f: (X, c_1, c_2) \to (Y, c_1^*, c_2^*)$ is fuzzy bicontinuous iff $c_1 f^{-1}(B) \subseteq f^{-1}(c_2^* B)$ for every $B \subseteq Y$.

Proof : Let B be a non empty subset of X. Then $f^{-1}(B) \subseteq X$ Since f is fuzzy bicontinuous, $f(c_1 f^{-1}(B)) \subseteq c_2^* f(f^{-1}B) \subseteq c_2^* B$. Therefore, $c_1 f^{-1}(B) \subseteq f^{-1}(c_2^* B)$.

Conversely : Let A be a non empty subset of Y Then $f(A) \subseteq Y$. Thus $c_1 f^{-1}(f(A)) \subseteq f^{-1}(c_2^* f(A))$. Consequently, $f(c_1 A) \subseteq f(c_1 f^{-1}(f(A)) \subseteq f(f^{-1}(c_2^* f(A)) \subseteq c_2^* f(A)$. Hence, f is fuzzy bicontinuous.

Proposition 5.2 : Let (X, c_1, c_2), (Y, c_1^*, c_2^*) and (Z, c_1^{**}, c_2^{**}) are fuzzy biclosure spaces. If mapping $f : (X, c_1, c_2) \rightarrow (Y, c_1^*, c_2^*)$ is fuzzy bicontinuous and $g: (Y, c_1^*, c_2^*) \rightarrow (Z, c_1^{**}, c_2^{**})$ is 2- continuous, then $gof: (X, c_1, c_2) \rightarrow (Z, c_1^{**}, c_2^{**})$ is fuzzy bicontinuous.

Proof : Let A be a non empty subset of X Since $gof(c_1 A) = g(f(c_1 A))$ and f fuzzy bicontinuous, $g(f(c_1 A)) \subseteq g(c_2^* f(A))$. Since g is 2-continuous $g(c_2^* f(A)) \subseteq c_2^{**} g(f(A))$. Thus $gof(c_1 A) \subseteq c_2^{**} g(f(A))$. Consequently, gof is fuzzy bicontinuous.

6- *Strong fuzzy pairwise continuous Mappings :*

Definition 6.1: Let (X, c) and $(X, c*)$ be fuzzy closure spaces and A map $f: (X, c) \rightarrow (Y, c^*)$ is called fuzzy strong continuous map if $f(c(A)) \subset f(A) \quad \forall A \subset X$.

Remark : It can be easily seen that fuzzy strong pairwise continuous map $\Rightarrow$ fuzzy continuous map.

Definition 6.2 : Let (X, c_1, c_2) and (Y, c_1^*, c_2^*) be fuzzy biclosure spaces. Then mapping $f : (X, c_1, c_2) \rightarrow (Y, c_1^*, c_2^*)$ is called strong fuzzy pairwise continuous map iff $f(c_i(A)) \subseteq f(A)$ i.e. $f(c_i(A))(x) \leq (f(A))(x) \quad \forall A \subset X$.

Theorem 6.1: A mapping $f: (X, c_1, c_2) \rightarrow (Y, c_1^*, c_2^*)$ is strongly fuzzy pairwise continuous map iff $f^{-1}(B)$ is $c_i -$ closed for all $B \subset Y$, where $i = 1{,}2$.

Proof : Let $f : (X, c_1, c_2) \rightarrow (Y, c_1^*, c_2^*)$ be strongly fuzzy pairwise continuous map then $f(c_i(A)) \subset f(A)$.

If $B \subset Y$ then let, $x_r \in c_i(f^{-1}(B))$

And $f(f^{-1}(B)) \subset B$ $\quad\quad \{\because f(f^{-1}(B)) \subset f(c_i(f^{-1}(B))$

By definition of fuzzy continuity $f(c_i(f^{-1}(B)) \subset f(f^{-1}(B)) = B$

i.e. $f(c_i(f^{-1}(B)) \subset B$

$\therefore f(x_r) \subset B$. Then $x_r \in f^{-1}(B)$

$\therefore c_i(f^{-1}(B)) \subset f^{-1}(B)$

But we always have-

$f^{-1}(B) \subset c_i(f^{-1}(B))$

$\therefore c_i(f^{-1}(B)) = f^{-1}(B), \ \forall B \subset Y$.

$\Rightarrow f^{-1}(B)$ is $c_i -$ closed.

Conversely : Let $A \subset X$. Then $A \subset f^{-1}(f(A))$ and $c_i(A) \subset c_i(f^{-1}(f(A)) = f^{-1}(f(A))$

$\{\because f^{-1}(f(A))$ is c_i-closed.$\}$

$\therefore\ f(c_i(A)) \subset f\left(f^{-1}\left(f(A)\right)\right)$

$\Rightarrow\ f(c_i(A)) \subset \left(f(A)\right)$

$\Rightarrow f$ is strong fuzzy pairwise continuous.

Theorem 6.2 : Composite of strong fuzzy pairwise continuous functions is also strong fuzzy pairwise continuous.

Proof : The proof is on parallel lines as in proposition 5.2. So is omitted.

Conclusion

In this paper we have introduced and studied various types of the concepts of continuity in fuzzy biclosure space viz. fuzzy continuous mapping, fuzzy bicontinuous mapping and fuzzy strongly pairwise continuous mapping. We have obtained some interesting results related to these continuity in fuzzy biclosure spaces.

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