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A Generalized Subclass of Multivalent Functions Related to Sigmoid Function

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Abstract

In this paper, the authors investigate the initial coefficient bounds for a new generalized subclass of multivalent functions related to Sigmoid function. Also the relevant connections to Fekete-Szegö inequality and Hankel determinant for these classes are briefly discussed. Our results serve as a new generalization in this direction.

Key words: Analytic functions, Starlike function, Convex function, Alpha-convex function, Subordination, Sigmoid function, Fekete- Szegö problem.

Mathematics Subject Classification: 30C45, 33E99

1. Introduction and Preliminaries

The theory of special functions is significantly important to scientists and engineers. Though not with any specific definition but its applications extend to physics, computer etc. Recently, the theory of special functions has been overshadowed by other fields like real analysis, functional analysis, algebra, topology and differential equations.

There are various special functions but we shall concern with one of the activation function known as sigmoid function or simple logistic function. Activation function is an information process that is inspired by the biological nervous system such as brain processes information. It comprises of large number of highly interconnected processing elements (neurons) working together to solve a specific task. The function works in similar way the brain does, it learns by examples and cannot be programmed to solve a specific task.

The sigmoid function of the form

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$$h(z)=\frac{1}{1+e^{-z}} \tag{1.1}$$

is differentiable and has the following properties:

- It outputs real numbers between 0 and 1.
- It maps a very large input domain to a small range of outputs.
- It never loses information because it is a one-to-one function.
- It increases monotonically.

The four properties above shows that sigmoid function is very useful in geometric function theory.

Let A_p denote the class of functions of the form

$$f(z)=z^p+\sum_{n=2}^{\infty}a_{n+p}z^{n+p}, \tag{1.2}$$

which are analytic and *p*-valent in the open unit disc $E=\{z:|z|<1\}$.

Let U be the class of bounded functions

$$w(z)=\sum_{n=1}^{\infty}c_n z^n, \tag{1.3}$$

which are regular in the unit disc and satisfying the conditions

$$w(0)=0 \text{ and } |w(z)|<1 \text{ in } E.$$

For functions f and g analytic in *E*, we say that f is subordinate to g, denoted by $f\prec g$, if there exists a Schwarz function $w(z)\in U$, $w(z)$ analytic in E with $w(0)=0$ and $|w(z)|<1$ in *E*, such that $f(z)=g(w(z))$. It follows from Schwarz lemma that $f(z)\prec g(z)(z\in E)\Rightarrow f(0)=g(0)$ and $f(E)\subset g(E)$ (see detail in[5]).

If $f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}$ and $g(z)=z^p+\sum_{n=1}^{\infty}b_{n+p}z^{n+p}$ are in A_p, then Convolution or Hadamard product of the functions *f* and *g* is denoted by $f*g$ and is defined as

$$(f*g)(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}b_{n+p}z^{n+p}.$$

Let $\varphi(z)$ be an analytic function with positive real part in *E* such that $\varphi(0)=1$ and $\varphi'(0)>0$ and maps *E* onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Fekete and Szegö in 1933 gave the sharp bound for the functional $|a_3-\mu a_2^2|$ for the functions in the class *S* of univalent functions when μ is real. The determination of the sharp bounds for the functional $|a_3-\mu a_2^2|$ is known as the Fekete-Szegö problem and this has been investigated by several authors for different subclasses of univalent functions.

In 1976, Noonan and Thomas[9] stated the *qth* Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. Easily, one can observe that the Fekete and Szegö functional is $H_2(1)$.

For $q = 2$ and $n = 2$, $H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$ is the second Hankel determinant.

A function $f(z) \in A_p$ is said to be in the class $f(z) \in S^*_{b,p}(\varphi)$ if

$$1 + \frac{1}{b}\left[\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right] \prec \varphi(z)(p \in N, z \in E).$$

A function $f(z) \in A_p$ is said to be in the class $f(z) \in C_{b,p}(\varphi)$ if it satisfies

$$1 - \frac{1}{b} + \frac{1}{bp}\left[1 + \frac{zf''(z)}{f'(z)}\right] \prec \varphi(z)(p \in N, z \in E).$$

The classes $S^*_{b,p}(\varphi)$ and $C_{b,p}(\varphi)$ were studied in[1]. For $b = 1$ we have the classes $S^*_p(\varphi)$ and $C_p(\varphi)$ (see[2]) and for $p = b = 1$ the classes reduced to the classes $S^*(\varphi)$ and $C(\varphi)$ which were earlier introduced and investigated in[4]. These classes become the classes of starlike and convex functions respectively when

$$\varphi(z) = \frac{1+z}{1-z}.$$

Also for $p = 1$ and $\varphi(z) = \frac{1+z}{1-z}$, the classes $S^*_{b,p}(\varphi)$ and $C_{b,p}(\varphi)$ reduces to the classes $S^*(b)$ and $C(b)$ which were investigated in[8] and [14].

A function $f(z) \in A_p$ is said to be in the class $f(z) \in M_p(\alpha)$ if for $\alpha \geq 0$,

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0.$$

The function in class $M_p(\alpha)$ are called *p*-valent alpha-convex function. Obviously $M_1(\alpha) \equiv M(\alpha)$, the class of alpha-convex functions introduced by Mocanu[6].

Motivated by above defined classes, we introduce the following generalized subclasses of *p*-valent analytic functions of complex order related to sigmoid functions.

For $b \in C$, let the class $T_{n,p}(\alpha; b, \Phi_{m,n})$ denote the subclass of A_p consisting of functions of the form (1.2) and satisfying the following condition

$$p + \frac{1}{b}\left[(1-\alpha)\frac{z\left(D_{n-1}^{p} f(z)\right)'}{D_{n-1}^{p} f(z)} + \alpha\,\frac{z\left(D_{n}^{p} f(z)\right)'}{D_{n}^{p} f(z)} - p\right] > 0 \,,$$

for $0 \le \alpha \le 1$ and $\Phi_{m,n}(z)$ is a simple logistic sigmoid activation function and

$$D_{n-1}^{p} f(z) = \frac{z^{p}}{(1-z)^{n+p}} * f(z) \text{ where } n + p > 0.$$

The following observations are obvious:

(i) $T_{0,p}(\alpha; b, \Phi_{m,n}) \equiv M_p(\alpha; b, \Phi_{m,n})$.

(ii) $T_{0,p}(0; b, \Phi_{m,n}) \equiv S_{b,p}^{*}(\Phi_{m,n})$.

(iii) $T_{0,p}(1; b, \Phi_{m,n}) \equiv C_{b,p}(\Phi_{m,n})$.

Recently, various authors as Abiodun[10], Murugusundramoorthy *et al.*[7], Olatunji *et al.*[12], and Olatunji[11] have studied sigmoid function for different classes of analytic and univalent functions.

In the present work, we obtained few coefficient bounds for the class $T_{n,p}(\alpha; b, \Phi_{m,n})$ and the relevant connection with Fekete-Szegö theorems and Hankel determinant.

To prove our result we shall make use of the following lemmas:

Lemma 1.[13] If a function $p \in P$ is given by

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots (z \in E),$$

then $|p_k| \le 2, k \in N$ where *P* is the family of all functions analytic in *E* for which $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0 (z \in E)$.

Lemma 2.[3] Let h be the sigmoid function defined in (1.1) and

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m,$$

then $\Phi(z) \in P, |z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 3.[3] Let

$$\Phi_{m,n}(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m,$$

then $\left|\Phi_{m,n}(z)\right| < 2.$

Lemma 4.[3] If

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

where $c_n = \dfrac{(-1)^{n+1}}{2n!}$, then $|c_n| \le 2, n \in N$ and the result is sharp for each *n*.

2. Initial Coefficients

Theorem 2.1 If $f(z) \in A_p$ of the form (1.2) is belonging to $T_{n,p}(\alpha; b, \Phi_{m,n})$, then

$$|a_{p+1}| \le \frac{p|b|}{2[n+p+\alpha]}, \tag{2.1}$$

$$|a_{p+2}| \le \frac{p^2|b|^2\left[(n+p)^2 + \alpha(2n+2p+1)\right]}{4(n+p+\alpha)^2(n+p+1)(n+p+2\alpha)} \tag{2.2}$$

and

$$|a_{p+3}| \le |h_1| \tag{2.3}$$

where

$$h_1 = \frac{p^3 b^3}{8(n+p+\alpha)^3(n+p+1)(n+p+2)(n+p+3\alpha)} \left[\begin{array}{l} \dfrac{3\{(n+p)^2 + \alpha(2n+2p+1)\}\{(n+p)^2(n+p+\alpha+1) + \alpha(n+p+2)(2n+2p+1)\}}{2} \\ -\{(n+p)^3 + 3\alpha(n+p+1) + \alpha\} \end{array} \right].$$

Proof. As $f(z) \in T_{n,p}(\lambda; b, \Phi_{m,n})$, therefore

$$p + \frac{1}{b}\left[(1-\lambda)\frac{z\left(D_{n-1}^p f(z)\right)'}{D_{n-1}^p f(z)} + \lambda \frac{z\left(D_n^p f(z)\right)'}{D_n^p f(z)} - p\right] = p\Phi_{m,n}(z) \tag{2.4}$$

where $$\Phi_{m,n}(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 - \dots \tag{2.6}$$

We have

$$\frac{z\left(D_{n-1}^p f(z)\right)'}{D_{n-1}^p f(z)} = p + (n+p)a_{p+1}z + (n+p)\left[(n+p+1)a_{p+2} - (n+p)a_{p+1}^2\right]z^2 + \dots \tag{2.7}$$

Replacing n by $n+1$ in (2.7), we get

$$\frac{z\left(D_n^p f(z)\right)'}{D_n^p f(z)} = p + (n+p+1)a_{p+1}z + (n+p+1)\left[(n+p+2)a_{p+2} - (n+p+1)a_{p+1}^2\right]z^2 + \dots \tag{2.8}$$

Using (2.6), (2.7) and (2.8), (2.4) can be expanded as

$$(n+p+\alpha)a_{p+1}z+\left[(n+p+1)(n+p+2\alpha)a_{p+2}-\left((n+p)^2+\alpha(2n+2p+1)\right)a_{p+1}^2\right]z^2$$
$$+\left[\begin{array}{l}(n+p+1)(n+p+2)(n+p+3\alpha)a_{p+3}-\dfrac{3}{2}\left\{(n+p)^2(n+p+\alpha)+(n+p)^2+\alpha(n+p+2)(2n+2p+1)\right\}a_{p+1}a_{p+2}\\+\left\{(n+p)^3+3\alpha(n+p+1)+\alpha\right\}a_{p+1}^3\end{array}\right]z^3+\ldots \tag{2.9}$$
$$=bp\left[\frac{1}{2}z-\frac{1}{24}z^3+\ldots\right]\left[(1+\alpha(p-1))+(1+\alpha p)a_{p+1}z+(1+\alpha(p+1))a_{p+2}z^2+\ldots\right]$$

Equating the coefficients of z, z^2 and z^3 in (2.9), we obtain

$$a_{p+1}=\frac{pb}{2[n+p+\alpha]}, \tag{2.10}$$

$$a_{p+2}=\frac{p^2b^2\left[(n+p)^2+\alpha(2n+2p+1)\right]}{4(n+p+\alpha)^2(n+p+1)(n+p+2\alpha)} \tag{2.11}$$

and

$$a_{p+3}=h_1. \tag{2.12}$$

Results (2.1), (2.2) and (2.3) can be easily obtained from (2.10), (2.11) and (2.12) respectively.

For $n=0$ in Theorem 2.1, the following result is obvious:

Corollary 2.1 If $f(z)\in M_p(\alpha;b,\Phi_{m,n})$, then

$$|a_{p+1}|\le\frac{p|b|}{2(p+\alpha)}, \qquad |a_{p+2}|\le\frac{p^2|b|^2\left[p^2+\alpha(2p+1)\right]}{4(p+\alpha)^2(p+1)(p+2\alpha)}$$

and

$$|a_{p+3}|\le\frac{p^3|b|^3}{8(p+\alpha)^3(p+1)(p+2)(p+3\alpha)}\left|\frac{3\left\{p^2+\alpha(2p+1)\right\}\left\{p^2(p+\alpha+1)+\alpha(p+2)(2p+1)\right\}}{2}-\left\{p^3+3(p+1)+\alpha\right\}\right|.$$

For $n=0$, $\alpha=0$ Theorem 2.1 gives the following result:

Corollary 2.2 If $f(z)\in S^*_{b,p}(\Phi_{m,n})$, then

$$|a_{p+1}|\le\frac{|b|}{2}, \quad |a_{p+2}|\le\frac{p^2|b|^2}{4(p+1)(p+2\alpha)} \text{ and}$$

$$|a_{p+3}|\le\frac{|b|^3}{8p(p+1)(p+2)}\left|\frac{3\left\{p^4(p+1)\right\}}{2}-\left\{p^3+3(p+1)\right\}\right|$$

Putting $n = 0,\ \alpha = 1$ in Theorem 2.1, we obtain the following result:

Corollary 2.3 If $f(z) \in C_{b,p}(\Phi_{m,n})$, then

$$|a_{p+1}| \le \frac{p|b|}{2(p+1)}, \qquad |a_{p+2}| \le \frac{p^2|b|^2}{4(p+1)(p+2)}$$

and

$$|a_{p+3}| \le \frac{p^3|b|^3}{8(p+1)^4(p+2)(p+3)}\left|\frac{3\{(p+1)^4(p+2)\}}{2} - \{p^3 + 3(p+1) + 1\}\right|.$$

3. Fekete-Szegö Inequality

Theorem. 3.1 If $f(z) \in A_p$ of the form (1.2) is belonging to $T_{n,p}(\alpha; b, \Phi_{m,n})$, then

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{p^2|b|^2}{4(n+p+\alpha)^2}\left|\frac{(n+p)^2 + \alpha(2n+2p+1)}{(n+p+1)(n+p+2\alpha)} - \mu\right|. \tag{3.1}$$

Proof. From (2.10) and (2.11), we have

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p^2 b^2}{4(n+p+\alpha)^2}\left[\frac{(n+p)^2 + \alpha(2n+2p+1)}{(n+p+1)(n+p+2\alpha)} - \mu\right]. \tag{3.2}$$

Hence (3.1) can be easily obtained from (3.2)

For $p = 1$, Theorem 3.1 gives the following result:

Corollary 3.1 If $f(z) \in M_\lambda(b, \Phi_{m,n})$, then

$$|a_3 - \mu a_2^2| \le \frac{|b|^2}{4(n+\alpha+1)^2}\left|\frac{(n+1)^2 + \alpha(2n+3)}{(n+2)(n+2\alpha+1)} - \mu\right|.$$

4. Second Hankel Determinant :

Theorem. 4.1 : If $f(z) \in A_p$ of the form (1.2) is belonging to $T_{n,p}(\alpha; b, \Phi_{m,n})$, then

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| \le \left|\frac{pbh_2}{2(n+p+\alpha)} - \mu h_1^2\right| \tag{4.1}$$

where h_1 and h_2, are defined in (2.10) and (2.11) respectively.

Proof. From (2.10), (2.11) and (2.12), we have

$$a_{p+1}a_{p+3} - \mu a_{p+2}^2 = \frac{pbh_2}{2(n+p+\alpha)} - \mu h_1^2. \tag{4.2}$$

Hence (4.1) can be easily obtained from (4.2)

For $p = 1$, Theorem 4.2 gives the following result:

Corollary 4.1 If $f(z) \in G_{\lambda}(b, \Phi_{m,n})$, then

5. Conclusion

The results obtained above serve as a new generalization of subclasses of multivalent functions related to Sigmoid functions. The investigation of initial coefficients bounds, Fekete-Szegö inequality and Hankel determinant for similar classes can be the scope of future research in this direction.

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