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A Common Fixed Point Theorem in Fuzzy Metric Space Using Common E. A. Like Property

UDAY DOLAS

Department of Mathematics, C.S.A.P.G. College
Sehore-466001, M.P. (India)
Email: udolas@gmail.com

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Abstract

In this paper, we are proving a common fixed point theorem for mappings satisfying common E.A. like property in fuzzy metric space. We generalize the result of Jain *et al.*[11], using rational inequality.

Key word: Fuzzy metric space, common E.A. like property and weak compatible maps.

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1. Introduction

In 1965, Zadeh[22] introduced the concept of fuzzy set. Following the concept of fuzzy sets Kramosil and Michalek[13] introduced the concept of fuzzy metric space in 1975. George and Veeramani[8] modified the notion of fuzzy metric spaces with the help of continuous t-norm, which shows a new way for further development of analysis in such spaces. It has been seen that the study of Kramosil and Michalek[13] of fuzzy metric space covered almost all the points in the way for developing this theory to the field of fixed point theorem, in particular for the study of contractive type maps. They have also shown that every metric induces a fuzzy metric. Singh[21] proved various fixed point theorems using the concepts of semi-compatibility, compatibility and implicit relations in Fuzzy metric space. Kumar and Pant[15] have given a common fixed point theorem for two pairs of compatible mapping satisfying expansion type condition in probabilistic Menger space. Recently, Jain *et al.*[11] improved the result of Kumar and Pant[15] by dropping the condition of continuity of the mapping and using semi and weak compatibility of the mapping in place of compatibility[1-7].

In this paper we prove common fixed point theorems for mappings satisfying common E.A. like property in fuzzy metric space, which generalize the result of Jain *et al.*[11] using rational inequality[12].

2. Preliminaries

Definition 2.1[18] A binary operation $*$: $[0,1]*[0,1] \rightarrow [0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that a $*$ b $\leq$ c $*$ d whenever a $\leq$ c and b $\leq$ d for all a,b,c,d $\in$ [0,1]. Example of t-norm are a $*$ b=ab and a $*$ a=min {a, b}.

Definition 2.2[13] The 3-tuple $(X, M, *)$ is said to be a Fuzzy metric space if X is an arbitrary set,* is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$,

(FM-5) $M(x, y, .) : [0,\infty) \rightarrow [0,1]$ is left continuous.

Where $M(x, y, t)$ can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$.

Definition 2.3.[9] Letbe a fuzzy metric space:

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, (denoted by $\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(2) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.

(3) A Fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Let $(X, M, *)$ be a fuzzy metric space with following condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.

Definition 2.4[19] A function M is continuous in Fuzzy metric space if and only if whenever $x_n \to x, y_n \to y$, then $\lim_{t\to\infty} M(x_n, y_n, t)$ $= M(x, y, t)$ for all $t > 0$.

Definition 2.5[14] Let A and B be mappings from a Fuzzy metric space $(X, M, *)$ into itself. The mappings A and B are said to be weakly compatible if they commute at their coincidence points, *i.e.* $Ax = Bx$ implies $ABx = BAx$.

Definition 2.6[21] Suppose A and S be two maps from a Fuzzy metric space $(X, M, *)$ into itself. Then they are said to be semi-compatible if $\lim_{n\to\infty} ASx_n = Sx$ whenever $\{x_n\}$ is a sequence that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$ $= x \in X$.

In[9] Grabiec has given two important lemmas for contraction condition. We have the following lemmas for expansion type condition.

Lemma 2.1 Let $\{x_n\}$ be a sequence in a Fuzzy metric space $(X, M, *)$ with (FM-6). If there exists a number $h > 1$ such that $M(x_{n+1}, x_n, ht) \le M(x_{n+2}, x_{n+1}, t)$ for all $t > 0$ and $n = 1, 2, 3, ...$ Then $\{x_n\}$ is Cauchy sequence in X.

Lemma 2.2 If for all $x, y \in X, t > 0$ and for a number $h > 1$, $M(x, y, ht) \le M(x, y, t)$ then $x = y$ Jain *et al.*[11] proved the following result.

Theorem 2.3 Let (X, M, *) be a complete Fuzzy metric space where * is continuous t-norm and satisfies $x * x \ge x$ for all $x \in [0, 1]$. Let A, B, S and T be self mappings of a Fuzzy metric space satisfying the following conditions[10,12]:

(3.1) A and B are surjective.

(3.2) (A, S) is semi-compatible and (B, T) is weakly compatible.

(3.3) $M(Au, Bv, hx) \le M(Su, Tv, x)$ for all $u, v \in X$ and $h > 1$.

Then A, B, S and T have a unique common fixed point in X.

3. Main Result

Theorem 3.1: Let (X, M, *) be a complete Fuzzy metric space where * is continuous t-norm and satisfies $t * t \ge t$ for all $t \in [0, 1]$. Let A, B, S and T be self mappings of a Fuzzy metric space satisfying the following conditions:

(3.1.1) $\forall$ x, $y \in X$, $t > 0$ and $h > 1$,

$$M(Ax, By, ht) \le \min\left\{M(Sx, Ax, t), M(Ty, By, t), \frac{rM(Sx, By, t) + sM(Sx, Ty, t)}{rM(By, Ty, t) + s}\right\},$$

Where r, s ≥ 0 with r&s cannot be simultaneously 0,

(3.1.2) Pairs (A, S) and (B, T) satisfy common E.A. like property.

(3.1.3) Pairs (A, S) and (B, T) are weakly compatible.

Then A, B, S and T have a unique common fixed point in X[16,17].

Proof: Since (A, S) and (B, T) satisfy common E. A. Like property therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$

where $z \in S(X) \cap T(X)$ or $z \in A(X) \cap B(X)$.

Suppose $z \in S(X) \cap T(X)$, now we have $\lim_{n\to\infty} Ax_n = z \in S(X)$ then $z = Su$ for some $u \in X$. No we claim that Au=Su, from (3.1.1) we have,

$$M(Au, By_n, kt) \le \min\left\{M(Su, Au, t), M(Ty_n, By_n, t), \frac{rM(Su, By_n, t) + sM(Su, Ty_n, t)}{rM(By_n, Ty_n, t) + s}\right\}$$

Taking limit $n \to \infty$, we get

$$M(Au, By_n, kt) \le \min\left\{M(z, Au, t), M(z, z, t), \frac{rM(z, z, t) + sM(z, z, t)}{rM(z, z, t) + s}\right\}$$

$$M(Au, z, kt) \leq \min\{M(z, Au, t), 1, 1\}$$
$$M(Au, z, kt) \leq M(z, Au, t)$$
$$M(Au, z, kt) \leq M(z, Au, t)$$

Lemma 2.2 implies that Au=z=Su.

Since the pair (A, S) is weak compatible, therefore Az = ASu = SAu = Sz.

Again, $\lim_{n\to\infty} By_n = z \in T(X)$ then z= Tv for some $v \in X$.

No we claim that Tv=Bv, from (3.1.1) we have,

$$M(Ax_n, Bv, kt) \leq \min\left\{M(Sx_n, Ax_n, t), M(Tv, Bv, t), \frac{rM(Sx_n, Bv, t) + sM(Sx_n, Tv, t)}{rM(Bv, Tv, t) + s}\right\}$$

Taking limit $n \to \infty$, we get

$$M(z, Bv, kt) \leq \min\left\{M(z, z, t), M(z, Bv, t), \frac{M(z, Bv, t) + M(z, z, t)}{M(Bv, z, t) + 1}\right\}$$

$$M(z, Bv, kt) \leq \min\{1, M(z, Bv, t), 1\}$$
$$M(z, Bv, kt) \leq M(z, Bv, t)$$
$$M(z, Bz, kt) \leq M(z, Bv, t)$$

Lemma 2.2 implies that Bv = z = Tv.

Since the pair (B, T) is weak compatible, therefore Tz = TBv = BTv = Bz.

Now we show that Az=z, from (3.1.1) we have,

$$M(Az, By_n, kt) \leq \min\left\{M(Sz, Az, t), M(Ty_n, By_n, t), \frac{rM(Sz, By_n, t) + sM(Sz, Ty_n, t)}{rM(By_n, Ty_n, t) + s}\right\}$$

Taking limit $n \to \infty$, we get

$$M(Az, z, kt) \leq \min\left\{M(Az, Az, t), M(z, z, t), \frac{rM(Az, z, t) + sM(Az, z, t)}{rM(z, z, t) + s}\right\}$$

$$M(Az, z, kt) \leq \min\{1, 1, M(Az, z, t)\}$$
$$M(Az, z, kt) \leq M(Az, z, t)$$
$$M(Az, z, kt) \leq M(Az, z, t)$$

Lemma 2.2 implies that Az = z.

Now we show that Bz=z, from (3.1.1) we have,

$$M(Ax_n, Bz, kt) \leq \min\left\{M(Sx_n, Ax_n, t), M(Tz, Bz, t), \frac{rM(Sx_n, Bz, t) + sM(Sx_n, Tz, t)}{rM(Bz, Tz, t) + s}\right\}$$

Taking limit $n \to \infty$, we get

$$M(z, Bz, kt) \leq \min\left\{M(z, z, t), M(Bz, Bz, t), \frac{rM(z, Bz, t) + sM(z, Bz, t)}{rM(Bz, Bz, t) + s}\right\}$$

$$M(z, Bz, kt) \leq \min\{1, 1, M(z, Bz, t)\}$$
$$M(z, Bz, kt) \leq M(z, Bz, t),$$
$$M(z, Bz, kt) \leq M(z, Bz, t)$$

Lemma 2.2 implies that Bz = z.

Hence, $Az = Sz = Bz = Tz = z$.

Thus is common fixed point of A, B, S and T.

To prove uniqueness we suppose that p and q are two common fixed point of A, B, S and T such that p ≠ q, then from (3.1.1) we have[20],

$$M(Ap, Bq, kt) \leq \min\left\{M(Sp, Ap, t), M(Tq, Bq, t), \frac{rM(Sp, Bq, t) + sM(Sp, Tq, t)}{rM(Bq, Tq, t) + s}\right\}$$

$$M(p, q, kt) \leq \min\left\{M(p, p, t), M(q, q, t), \frac{rM(p, q, t) + sM(p, q, t)}{rM(q, q, t) + s}\right\}$$

$$M(p, q, kt) \leq \min\{1, 1, M(p, q, t)\}$$

$$M(p, q, kt) \leq M(p, q, t),$$

$$M(p, q, kt) \leq M(p, q, t)$$

Lemma 2.2 implies that p = q. This completes the proof of the theorem.

Remark 3.2: Theorem 3.1 not requires the completeness of the space. Replacing semi-compatible mapping by common E.A. like property we have generalized the result of Jain *et al.*[11] using rational inequality.

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