Ultra Scientist Vol. 26(2)A, 141-148 (2014). ISSN 2231-3478 (Print)
www.ultrascientist.org 2319-8052 (Online)

On Generalized Nörlund Summability Factors of Infinite Series

SATISH CHANDRA and SAAJAN KUMAR

Department of Mathematics S.M. Post – Graduate College,
Chandausi – 244412 (INDIA)

(Acceptance Date25th May, 2014)

Abstract

In this paper we have proved a theorem on generalized Nörlund summability Factors of infinite series , which generalizes various known results. However our theorem is as follows :

Theorem : Let $\{p_n\}$ be a non-negative and non-increasing, $\{x_n\}$ is a positive non-decreasing sequence and $\{\lambda_n\}$ is a positive decreasing sequence such that

$$\sum_{\nu=1}^{n} \lambda_\nu \mid s_\nu \mid = \mathbf{O}(x_n) \quad \text{as } n \to \infty$$

$$n\,\Delta\lambda_n = \mathbf{O}(\lambda_n)$$

$$\sum_{n=\nu+1}^{\infty} \frac{1}{\tau_{n-1}} \left(\frac{p_n}{\tau_n} \right) = \mathbf{O}\left(\frac{1}{\tau_\nu} \right)$$

$$\sum_{\nu=1}^{\infty} \mid \epsilon_\nu \mid \mid x_\nu \mid < \infty$$

Then the series $\sum a_n \epsilon_n \lambda_n$ is summable by $G(N, p, \lambda)$

Key words and phrases : Generalized Nörlund summability, Infinite series.

2000 Mathematics subject classification : 40D05 , 40E05 , 40F05 and 40G05.

1. Definitions and Notations :

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with s_n is its n^{th} partial sum The (N,P,λ) transforem * of

$$s_n = \sum_{\nu=0}^{n} a_\nu \text{ is defined by}$$

$$\tau_n = \frac{\sum_{\nu=0}^{n} p_{n-\nu}\,\lambda_\nu s_\nu}{r_n}$$

where

$$r_n = \sum_{\nu=0}^{n} p_n\,\lambda_{n-\nu} \qquad (p_{-1} = \lambda_{-1} = r_{-1} = 0)$$

$$\neq 0 \qquad \text{for } n \geq 0$$

The series $\sum_{n=0}^{\infty} a_n$ or the sequence $\{s_n\}$ is said to be summable (N,p,λ) to s, if $\tau_n \to s$ as $n \to \infty$ and is said to be absolutely summable $|N,p,\lambda|$ if $|\tau_n| \in BV$ and when this happens, we shall write symbolically by $\{s_n\} \in |N,p,\lambda|$.

The method (N,p,λ) reduces to the method[5] (N,p_n) when $\lambda_n=1$;(Hardy p.64]; to Euler-knopp method (E,δ) when[5]

$$p_n = \frac{\alpha^n \delta^n}{n!}, \lambda_n = \frac{\alpha^n}{n!} \ (\alpha > 0, \delta > 0)$$

(Hardy p. 178) ; to the method (C,α,β)(Borwein[1,2]) when $p_n = \binom{n+\alpha+1}{\alpha}$, $\lambda_n = \binom{n+\beta}{\beta}$.

*Borwein[1] This is called the generalized Nörlund transform.

We write

$$\epsilon_n = p_n - p_{n-1} = \Delta p_n$$

$$\mu_n = q_n - q_{n-1} = \Delta q_n$$

and

$$\xi_n = \delta_n^\alpha$$

furthermore $\delta_n = \sum_{\nu=0}^{n} \lambda_\nu$

we note that

$$r_n = \sum_{\nu=0}^{n} p_{n-\nu}\lambda_\nu = \sum_{\nu=0}^{n} \epsilon_{n-\nu}\,\delta_\nu$$

and

$$\sum_{\nu=0}^{n} p_{n-\nu}\lambda_\nu s_\nu = \sum_{\nu=0}^{n} (p_{n-\nu} - p_{n-\nu-1}) \sum_{j=0}^{\nu} \lambda_j s_j$$

$$= \sum_{\nu=0}^{n} \epsilon_{n-\nu}\, t_\nu\, s_\nu$$

where

$$t_\nu = \frac{1}{\delta_\nu} \sum_{i=0}^{\nu} \lambda_i s_i$$

$$= \frac{1}{\delta_\nu} \sum_{i=0}^{\nu} (\delta_i - \delta_{i-1})\, a_i$$

Here $\{t_\nu\}$ is the $(\overline{N},\lambda)$ mean[5] (Hardy p. 5) which is equivalent to $(R^*,\delta_{n-1}, 1)$ mean[5] (Hardy p 1.13).

Rewriting τ_n is terms of the simplification given above, we now have

$$\tau_n = \frac{\sum_{\nu=0}^{n} (p_{n-\nu} - p_{n-\nu-1})\, t_\nu \delta_\nu}{\sum_{\nu=0}^{n} (p_{n-\nu} - p_{n-\nu-1})\, \delta_\nu}$$

and this form suggests that we can have the following extension of the (N, p, λ) method.

We now write, for any $\{\epsilon_n\}$

$$(1.1) \qquad \tau_n^{(\alpha)} = \frac{\sum_{\nu=0}^{n} \epsilon_{n-\nu}\, t_\nu^\alpha\, \delta_\nu^\alpha}{\sum_{\nu=0}^{n} \epsilon_{n-\nu}\, \delta_\nu^\alpha}$$

$$= \frac{\sum_{\nu=0}^{n} \epsilon_{n-\nu}\, t_\nu^\alpha\, \xi_\nu}{\sum_{\nu=0}^{n} \epsilon_{n-\nu}\, \xi_\nu}$$

where

$$t_n^{(\alpha)} = \frac{1}{\delta_n^\alpha} \sum_{\nu=0}^{n} (\delta_\nu - \delta_{\nu-1})^\alpha\, a_\nu$$

We denote this mean by G $(N,p,\lambda)_\alpha$ (Dhal [4]).
When $\alpha = 1$

$\tau_n^{(1)}$ = (N,p,λ) (s_n), the G (N,p,λ) method reduces to (N,p,λ) method.

2. Dealing with the absolute Nörlund summability Sulaiman[7] has proved the following theorem.

Theorem A : Let $\{p_n\}$ be a sequence of positive numbers. Let $t_n{}^1$ denote the n^{th} (C, 1) mean of $\{n\, a_n\}$. If

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} |\epsilon_n|^k |t_n^1|^k < \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \left(\frac{P_n}{p_n}\right)^{k-1} |\epsilon_n|^k |t_n^1|^k < \infty$$

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta\epsilon_n|^k |t_n^1|^k < \infty$$

then the series $\sum a_n \epsilon_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

In 2003 Rhoades and Savas[6] have extended the theorem of Sulaiman in the following form.

Theorem B : Let $\{p_n\}$ be a positive sequence such that

$$\sum_{n=\nu+1}^{\infty} \frac{n^{k-1}}{P_{n-1}} \left(\frac{p_n}{P_n}\right)^k = \mathbf{O}\left(\frac{\nu^{k-1} p_\upsilon^{k-1}}{P_\nu^k}\right)$$

Let t_n^1 denote the n^{th} (C, 1) mean of {n an}. If

$$\sum_{\nu=1}^{\infty} \nu^{k-1} |\epsilon_\nu|^k |t_\nu^1|^k < \infty$$

$$\sum_{\nu=1}^{\infty} \nu^{k-1} |\Delta\epsilon_\nu|^k |t_\nu^1|^k < \infty$$

then the series $\sum a_n \epsilon_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

The aim of this paper is to generalize Theorem B for generalized Nörlund summability[3].

3. Main Theorem : We shall prove the following main theorem

Theorem : Let $\{p_n\}$ be a non-negative and non-increasing sequence, $\{x_n\}$ is a positive non decreasing sequence and $\{\lambda_n\}$ is a positive decreasing sequence such that

(3.1) $\sum_{\nu=1}^{n} \lambda_\nu |s_\nu| = O(x_n)$ as $n \to \infty$

(3.2) $n\,\Delta\lambda_n = O(\lambda_n)$

(3.3) $\sum_{n=\nu+1}^{\infty} \frac{1}{\tau_{n-1}} \left(\frac{p_n}{\tau_n}\right) = O(1)\left(\frac{1}{\tau_\nu}\right)$

and

(3.4) $\sum_{\nu=1}^{\infty} |\epsilon_\nu| \, |x_\nu| < \infty$

Then the series $\sum a_n \epsilon_n \lambda_n$ is summable by $G(N,p,\lambda)$.

4. Proof of the theorem :

$\tau_n^{(\alpha)}$ denote the $G(N,p,\lambda)$ mean of the series $\sum a_n \epsilon_n \lambda_n$. Then we have

$$\tau_n^{(\alpha)} - \tau_{n-1}^{(\alpha)} = \frac{1}{\tau_n}\sum_{\nu=0}^{n} \tau_{n-\nu}\, a_\nu \epsilon_\nu \lambda_\nu$$

$$- \frac{1}{\tau_{n-1}}\sum_{\nu=0}^{n-1} \tau_{n-1-\nu}\, a_\nu \epsilon_\nu \lambda_\nu$$

$$= \frac{1}{\tau_n \tau_{n-1}}\sum_{\nu=1}^{n} [\tau_\nu p_{n-\nu} - \tau_{n-\nu} p_n]\, a_\nu \epsilon_\nu \lambda_\nu$$

$$= \frac{p_n}{\tau_n\tau_{n-1}}\sum_{\nu=1}^{n}(\tau_\nu - \tau_{n-\nu})a_\nu \epsilon_\nu \lambda_\nu + \frac{1}{\tau_n\tau_{n-1}}\sum_{\nu=1}^{n}(p_{n-\nu} - p_n)a_\nu \epsilon_\nu \lambda_\nu$$

$$= \frac{p_n}{\tau_n\tau_{n-1}}\sum_{\nu=1}^{n-1}\Delta_\nu[(\tau_n - \tau_{n-\nu})\epsilon_\nu \lambda_\nu]\, s_\nu + \frac{p_n}{\tau_n\tau_{n-1}}(\tau_n - \tau_0)\tau_n \epsilon_n \lambda_n s_n$$

$$+ \frac{1}{\tau_{n-1}}\sum_{\nu=1}^{n-1}\Delta_\nu[(p_{n-\nu} - p_n)\epsilon_\nu \lambda_\nu]\, s_\nu + \frac{1}{\tau_{n-1}}(p_0 - p_n)\tau_n \epsilon_n \lambda_n s_n$$

So

$$|\tau_n^{(\alpha)} - \tau_{n-1}^{(\alpha)}| \le \frac{p_n}{\tau_n\tau_{n-1}}\sum_{\nu=1}^{n}\Delta_\nu[(\tau_n - \tau_{n-\nu})|\epsilon_\nu|\lambda_\nu]\,|s_\nu|$$

$$+ \frac{1}{\tau_{n-1}}\sum_{\nu=1}^{n}\Delta_\nu[(p_{n-\nu} - p_n)|\epsilon_\nu|\lambda_\nu]\,|s_\nu| + p_0|\epsilon_n|\lambda_n|s_n|$$

$$= t_n(1) + t_n(2) + t_n(3), \quad \text{say}$$

(4.1)
Now

$$\sum_{n=1}^{\infty} t_n(1) = \mathbf{O}(1) \sum_{n=1}^{\infty} \frac{p_n}{T_n T_{n-1}} \Bigg[\sum_{\nu=1}^{n} (\tau_n - \tau_{n-\nu}) p_{\nu+1} \mid \epsilon_\nu \mid \mid s_\nu \mid \lambda_\nu$$

$$+ \sum_{\nu=1}^{n} (\tau_n - \tau_{n-1}) \mid \Delta\epsilon_\nu \mid \mid s_\nu \mid \lambda_\nu + \sum_{\nu=1}^{n} (\tau_n - \tau_{n-\nu}) \mid \epsilon_\nu \mid \mid s_\nu \mid \Delta\lambda_\nu$$

$$+ \sum_{\nu=1}^{n} p_{n-\nu} \mid \epsilon_\nu \mid \mid s_\nu \mid \lambda_\nu \Bigg]$$

$$= \mathbf{O}(1) \sum_{\nu=1}^{\infty} p_{\nu+1} \mid \epsilon_\nu \mid \mid s_\nu \mid \lambda_\nu \sum_{n=\nu}^{\infty} \frac{p_n}{\tau_n \tau_{n-1}} (\tau_n - \tau_{n-\nu})$$

$$+ \mathbf{O}(1) \sum_{\nu=1}^{\infty} \mid \Delta\epsilon_\nu \mid \mid s_\nu \mid \lambda_\nu \sum_{n=\nu}^{\infty} \frac{p_n}{\tau_n \tau_{n-1}} (\tau_n - \tau_{n-\nu})$$

$$+ \mathbf{O}(1) \sum_{\nu=1}^{\infty} \mid \epsilon_\nu \mid \mid s_\nu \mid \Delta\lambda_\nu \sum_{n=\nu}^{\infty} \frac{p_n}{\tau_n \tau_{n-1}} (\tau_n - \tau_{n-\nu})$$

$$+ \mathbf{O}(1) \sum_{\nu=1}^{\infty} \mid \epsilon_\nu \mid \mid s_\nu \mid \lambda_\nu \sum_{n=\nu}^{\infty} \frac{p_n p_{n-\nu}}{\tau_n \tau_{n-1}}$$

$$= \mathbf{O}(1) \sum_{\nu=1}^{\infty} p_{\nu+1} \mid \epsilon_\nu \mid \mid s_\nu \mid \lambda_\nu + \mathbf{O}(1) \sum_{\nu=1}^{\infty} \mid \Delta\epsilon_\nu \mid \mid s_\nu \mid \lambda_\nu$$

$$+ \mathbf{O}(1) \sum_{\nu=1}^{\infty} \mid \epsilon_\nu \mid \mid s_\nu \mid \Delta\lambda_\nu + \mathbf{O}(1) \sum_{\nu=1}^{\infty} \frac{1}{\nu} \mid \epsilon_\nu \mid \mid s_\nu \mid \lambda_\nu$$

By (3.2) and (3.4)

$$= \mathbf{O}(1)\sum_{\nu=1}^{\infty} |\epsilon_\nu||s_\nu|\lambda_\nu + \mathbf{O}(1)\sum_{\nu=1}^{\infty}|\Delta\epsilon_\nu||s_\nu|\nu\lambda_\nu$$

$$+\mathbf{O}(1)\sum_{\nu=1}^{\infty}|\epsilon_\nu||s_\nu|\nu\Delta\lambda_\nu$$

as $\frac{\tau_n}{n} = \mathbf{O}(1)$ and $\{p_n\}$ is bounded .

(4.2) $= N_1 + N_2 + N_3$ (say)

Now

$$\sum_{\nu=1}^{\infty}|\epsilon_\nu||s_\nu|\lambda_\nu = \sum_{\nu=1}^{\infty}|\epsilon_\nu||\lambda_\nu s_\nu|$$

$$= \mathbf{O}(1)\sum_{\nu=1}^{n-1}|\Delta\epsilon_\nu| x_\nu + \mathbf{O}(1)|\epsilon_n| x_n + \mathbf{O}(1)$$

$$= \mathbf{O}(1), \text{as } n \to \infty \text{ by } (3.1) \text{ and } (3.4)$$

Thus $N_1 = O(1)$

Again

$$\sum_{\nu=1}^{n}|\Delta\epsilon_\nu|\nu|\lambda_\nu s_\nu| = \mathbf{O}(1) \qquad \text{same as } N_1$$

So that $N_2 = O(1)$

(4.4) Lastly $\sum_{\nu=1}^{n}|\epsilon_\nu||s_\nu|\nu\Delta\lambda_\nu = \mathbf{O}(1)$ same as $N_!$

(4.4) Thus $N_3 = O(1)$

$$\sum_{n=1}^{\infty} t_n(2) = \mathbf{O}(1)\sum_{n=1}^{\infty}\frac{1}{\tau_{n-1}}\left[\sum_{\nu=1}^{n}(p_{n-\nu} - p_n)p_{\nu+1}|\epsilon_\nu||s_\nu|\lambda_\nu\right.$$

$$+\sum_{\nu=1}^{n}(p_{n-\nu} - p_\nu)|\Delta\epsilon_\nu||s_\nu|\lambda_\nu + \sum_{\nu=1}^{n}(p_{n-\nu} - p_n)|\epsilon_\nu||s_\nu|\Delta\lambda_\nu$$

$$+\sum_{\nu=1}^{n}(p_{n-\nu}-p_{n-\nu-1})\mid\epsilon_\nu\mid\mid s_\nu\mid\lambda_\nu\Bigg]$$

$$=\ \mathbf{O}(1)\sum_{\nu=1}^{\infty}p_{\nu+1}\mid\epsilon_\nu\mid\mid s_\nu\mid\lambda_\nu\sum_{n=\nu}^{\infty}\frac{(p_{n-\nu}-p_n)}{\tau_{n-1}}$$

$$+\mathbf{O}(1)\sum_{\nu=1}^{\infty}\mid\Delta\epsilon_\nu\mid\mid s_\nu\mid\lambda_\nu\sum_{n=\nu}^{\infty}\frac{(p_{n-\nu}-p_n)}{\tau_{n-1}}$$

$$+\mathbf{O}(1)\sum_{\nu=1}^{\infty}\mid\epsilon_\nu\mid\mid s_\nu\mid\Delta\lambda_\nu\sum_{n=\nu}^{\infty}\frac{(p_{n-\nu}-p_n)}{\tau_{n-1}}$$

$$+\mathbf{O}(1)\sum_{\nu=1}^{\infty}\mid\epsilon_\nu\mid\mid s_\nu\mid\lambda_\nu\sum_{n=\nu}^{\infty}\frac{\Delta_n(p_{n-\nu-1})}{\tau_{n-1}}$$

$$=\mathbf{O}(1)\sum_{\nu=1}^{\infty}\mid\epsilon_\nu\mid\mid\lambda_\nu s_\nu\mid+\mathbf{O}(1)\sum_{\nu=1}^{\infty}\mid\Delta\epsilon_\nu\mid\nu\mid\lambda_\nu s_\nu\mid+\mathbf{O}(1)\sum_{\nu=1}^{\infty}\mid\epsilon_\nu\mid\nu\,\Delta\lambda_\nu\mid s_\nu\mid$$

$= N_1 + N_2 + N_3$

from (4.2) and noting that $\frac{T_n}{n}=\mathbf{O}(1)$ as $n\to\infty$, and N_1, N_2 and N_3 have already been treated, so that[8]

(4.5) $$\sum_{n=1}^{\infty}t_n(2)=\mathbf{O}(1)$$

Finally

$$\sum_{n=1}^{\infty}t_n(3)=\ \mathbf{O}(1)\sum_{n=1}^{\infty}p_0\mid\epsilon_n\mid\mid\lambda_n s_n\mid$$

(4.6) $= O(1)$ as it has already been treated

Combining (4.1 to 4.6) the proof of theorem is completed .

References

1. Borwein, D., On products of sequences, *Jour. London Math. Soc., 33*, 352–357 (1958).
2. Borwein, D., On strong and absolute summability, *Proc. Glasgow Math. Assoc., 4,* 122–139 (1960).
3. Das, G., On some methods of summability, *Quart. Jour. of Math. Oxford, 17,* 244 – 256 (1996).
4. Dhal, N. B., The generalized |N , p , q | summability, *Sambalpur University Journal,* Vol. *4,* 1–11 (1971).
5. Hardy, G. H., Divergent Series, Oxford University Press, Oxford (1949).
6. Rhoades, B. E., A summability type factor theorem, and Ekrem, S. Tamkang *Journal of Mathematics,* Vol. *34,* No. 4, 395– 401 (2003).
7. Sulaiman, W. T., A study on a relation between two summability methods, *Proc. Amer. Math. Soc. 115,* 303–312 (1992).
8. Zygmund, A., Trigonometric series, Vols I, II (II Ed.) Cambridge University Press, Cambridge (1956).