3-Minimally Nonouterplanar Graphs of Semitotal – Block Graphs and Total – Block Graphs

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Abstract

In this paper, we obtain characterizations of graphs whose semitotal-block graphs and total-block graphs are 3-minimally nonouterplanar.

1. Introduction

In Kulli introduced the concepts of the semitotal-block graphs and total-block graphs. In the planarity and outer planarity of these graph valued functions were discussed. In, one finds the minimally non-outer planarity of these graph valued functions. In, D.G Akka and M.S. Patil finds the 2-minimally non-outer planarity of these graph valued functions. In this paper, we obtain the characterizations of graphs whose semitotal-block graphs and total-block graphs are 3-minimally nonouterplanar.

The following definitions will be noted for later use. A graph G is called a block if it has more than one vertex, is connected and has no cutvertices.

A block of a graph G is a maximal subgraph of G which itself a block. If is a block of G, then we say that vertex and block B are incident with each other as are and B so on. If two distinct blocks and are incident with a common cutvertex, then they are adjacent blocks. The vertices and blocks of a graph are called the members.
The following will be useful in the proof of our results.

Lemma 1. For the graph \( K_{1,3}, i[T(K_{1,3})] = 2 \).

Theorem A. The total block graph \( T_B(G) \) of a connected outer planar graph \( G \) is 2-minimally nonouterplanar if and only if
1) \( G \) is a path \( P_n, n \geq 3 \) together with an end edge adjoined at some non-end vertex or
2) \( G \) is a path \( P_n, n \geq 2 \) together with two vertices each adjoined to a pair of adjacent vertices of \( P_n \) or
3) \( G \) is a cycle of length 4 together with two paths \( P_m \) and \( P_n (m \geq 1, n \geq 2) \) adjoined at two consecutive vertices or
4) \( G \) is a cycle \( C_n, n \geq 4 \) with a diagonal edge joining a pair of vertices of length exactly 2.

Theorem B. The total block graph \( T_B(G) \) of a graph \( G \) is planar if and only if \( G \) is outer planar and every cutvertex of \( G \) lies on at most 3 blocks.

Theorem C. The total block graph \( T_B(G) \) of a graph \( G \) is outer planar if and only if each component of \( G \) is a path.

Theorem D. A graph \( G \) is a cycle if and only if the semitotal – block graph and total-block graph are isomorphic to a wheel.

Theorem E. The total graph \( T(G) \) of a graph \( G \) is planar if and only if the maximum degree among the vertices of \( G \) is at most 3 and every vertex of degree 3 is a cutvertex.

Theorem F. A connected graph \( G \) is a tree if and only if the graph \( T(G) \) and \( T_B(G) \) are isomorphic.

2. Main Results:

A criterion for the semitotal block graph of a connected graph \( G \) to be 3-minimally nonouterplanar is given in the following theorem.

Theorem 1. The semitotal block graph \( T_b(G) \) of a connected graph \( G \) is 3-minimally nonouterplanar if and only if (1) or (2) holds.
1) \( G \) has exactly three cycles and each cycle is a block or
2) \( G \) is a cycle \( C_n (n \geq 6) \) together with a diagonal edge joining a pair of vertices of length \( (n-3) \).

Proof. Suppose \( T_b(G) \) is 3-minimally nonouterplanar. Then \( T_b(G) \) is planar.

We now consider the following cases.

Case 1. Assume \( G \) is a tree. Then every block of \( T_b(G) \) is a triangle. Hence \( T_b(G) \), is outer planar, a contradiction.

Case 2. Assume \( G \) is not a tree.

We consider the following subcases of case 2.

Subcase 2.1. Suppose \( G \) has four
cycles. Then we have following subcases of subcase 2.1.

**Subcase 2.1.1.** Assume each cycle is a block. Then each cycle in $T_b(G)$ gives a wheel. Hence, $i[T_b(G)] > 3$, a contradiction.

**Subcase 2.1.2.** Assume $G$ has two cycles $C_1$ and $C_2$ as blocks. Then the remaining block is isomorphic to $K_4 - x$. In $T_b(G)$, $C_1$ and $C_2$ gives wheels as $W_1$ and $W_2$, where as $i(K_4 - x) = 2$. Thus, $i[T_b(G)] > 3$, a contradiction.

**Subcase 2.1.3.** Assume $G$ has two cycles $C_1$ and $C_2$ as blocks, which are isomorphic to $(K_4 - x)$. Then in $T_b(G)$ $i(K_4 - x) = 2$. Hence, $i[T_b(G)] > 3$, a contradiction.

**Subcase 2.1.4.** Assume $G$ has four cycles as a block $B$, and remaining blocks are edges of $G$. Thus, $G$ is a maximal outer planar graph with 6 vertices. In $T_b(G)$ the block vertex $b$ is adjacent with each vertex of $B$. Thus $i[T_b(G)] > 3$, a contradiction.

**Subcase 2.2.** Suppose $G$ has three cycles. Then there exists two blocks $B_1$ and $B_2$ in which one block $B_1$ is a cycle and $B_2$ is isomorphic to $K_4 - x$ such that atleast three vertices of $K_4 - x$ are adjacent to atleast one block. In embedding of $T_b(G)$, $i[T_b(B_1)] = 1$ and $i[T_b(B_2)] > 2$. Hence $i[T_b(G)] > 3$, a contradiction.

**Subcase 2.3.** Suppose $G$ has two cycles. Then we have subcases of subcase 2.3.

**Subcase 2.3.1.** Assume each cycle is a block. Then each block and corresponding block vertices forms wheel in $T_b(G)$. Hence, $i[T_b(G)] < 3$, a contradiction.

**Subcase 2.3.2.** Assume $G$ has two cycle as a block. Then we consider the following subcases of subcase 2.3.2.

**Subcase 2.3.2.1.** Suppose $G$ is isomorphic to $K_4 - x$. Then $i[T_b(G)] < 3$, a contradiction.

**Subcase 2.3.2.2.** Suppose a vertex of $K_4 - x$ is adjacent to some blocks. Then the block vertex $b$ corresponds to $K_4 - x$ is adjacent to all vertices of $K_4 - x$. In embedding $T_b(G)$ in any plane, we have $i[T_b(G)] < 2$, a contradiction.

**Subcase 2.3.2.3.** Suppose two vertices of $K_4 - x$ are adjacent to some blocks. Then the block vertex $b$ corresponds to $(K_4 - x)$ is adjacent to all vertices of $K_4 - x$. In plane embedding $T_b(G)$ in any plane, we have $i[T_b(G)] < 3$, a contradiction.

**Subcase 2.3.2.4.** Suppose three vertices of $(K_4 - x)$ are adjacent to atleast one block. Then in $T_b(G)$ the edges joining the block vertex of $K_4 - x$ and all vertices of $K_4 - x$ generates the planar representation such that the block vertices of blocks which are adjacent to three vertices of $K_4 - x$ lies in the interior region of $T_b(G)$ with $i[T_b(G)] > 3$, a contradiction.

**Subcase 2.3.2.5.** Suppose each vertex of $K_4 - x$ is adjacent to atleast one block. Then the block vertex $b$ corresponds to $(K_4 - x)$ is adjacent to all vertices of $(K_4 - x)$. In plane embedding
of $T_b(G)$. We have $i[T_b(G)] > 4$, a contradiction.

Subcase 2.4. Suppose $G$ is unicyclic graph. Then $i[T_b(G)] < 3$, a contradiction.

Case 3. Assume $G$ is a cycles $C_n$ ($n \geq 6$). Then we have following subcases of case 3.

Subcase 3.1. Suppose $G$ is a cycle $C_n$ ($n \geq 6$) as a block, with diagonal edge joining a pair of vertices of length $(n-3)$. Then $G$ contains one more cycle $C'_n$ as a block, clearly $i[T_b(G)] > 3$, a contradiction.

Subcase 3.2. Suppose $G$ is a cycle $C_n$ ($n \geq 6$) as a block, together with diagonal edge joining a pair of vertices of length $(n-4)$. Then $i[T_b(G)] > 3$, a contradiction.

Subcase 3.3. Suppose $G$ is a cycle $C_n$ ($n \geq 6$) as a block, together with diagonal edge joining a pair of vertices of length $(n-2)$. Then $i[T_b(G)] < 3$, a contradiction.

Conversely, suppose (1) holds. Then $G$ has exactly 3 cycles and each cycle is a block. By Theorem D, $T_b(G)$ has exactly three wheels as blocks. We know that every wheel is a minimally nonouterplanar. Thus $i[T_b(G)] = 3$.

Suppose (2) holds, now we can make use of mathematical induction on $n$ of cycle $C_n$. Suppose $n = 6$. Then $G$ is a cycle $C_6$ with the vertices $\{v_1, v_2, \ldots, v_6\}$, together with diagonal edge $x$ joining a pair of vertices $v_1$ and $v_4$ of length 3. So that $G$ has two cycles $C'_4$ and $C''_4$ with the vertices $v_1, v_2, v_3, v_4, v_1$ and $v_1, v_4, v_5, v_6, v_1$ respectively. Since $C_p, P \geq 6$ is a block, let $b$ be a block vertex in $T_b(G)$ which is adjacent to all the vertices of $C_p, P \geq 6$. In planar embedding of $T_b[C_6]$. It is easy to see that the planar embedding of $T_b(G)$, either $v_2, v_3$ of cycle $C'_4$ or $v_5, v_6$ of cycle $C''_4$ together with a block vertex $b$ lie in the interior region of planar embedding. Hence, $T_b(G)$ is 3-minimally nonouterplanar. Assume that result is true for $n = k$. Then $G$ is a cycle of length $C_k$. Clearly $T_b(G)$ is $(K-3)$ – minimally nonouterplanar.

Suppose $n = k + 1$. Then $G$ is a cycle of length $C_{k+1}$. Then we have to prove that $T_b(G)$ is $(k-2)$ – minimally nonouterplanar.

Let $v_{k+1}$ be vertex on a cycle $C_{k+1}$. If we delete a vertex $v_{k+1}$ from a cycle $C_{k+1}$ by deleting the edges $e_k = (v_{k+1}, v_k)$ and $e_{k+1} = (v_{k+1}, v_1)$ which are incident with a vertex $v_{k+1}$, resulting a cycle of length $C_k$. By inductive hypothesis $T_b(C_k)$ is $(k-3)$ – minimally nonouterplanar. Now rejoining a vertex $v_{k+1}$ to a cycle $C_k$ by joining the edges $e_{k+1}$ and $e_k$, resulting a cycle of length $C_{k+1}$. It has two cycles $C_4$ with the vertices $v_1, v_2, v_3, v_4, v_1$ and $C'_4$ with the vertices $v_1, v_4, v_5, \ldots, v_k, v_{k+1}, v_1$. In $T_b[C_{k+1}]$ the block vertex $b$ corresponds to $C_{k+1}$ is adjacent to all the vertices of $C_{k+1}$. Such that $v_2, v_3, b$ lies in the interior region of planar embedding.
Hence, $T_b(G)$ has $[(k+1)-3]=(k-2)$ minimally nonouterplanar.

Hence the proof.

In the following theorem, we establish a criterion for the total – block graph of a connected graph to be 3-minimally nonouterplanar.

**Theorem 2.** The total – block graph $T_b(G)$ of a connected outer planar graph $G$ is 3-minimally nonouterplanar if and only if.

1) $G$ has exactly three triangles as blocks, such that atmost two blocks lie on a common cut vertex,

or

2) $G$ has exactly two cycles $C_3$ and $C_4$ as blocks,

or

3) $G$ is a triangle together with two paths $P_m$ and $P_n$ ($m \geq 2$, $n \geq 2$) incident at a same vertex,

or

4) $G$ is a triangle together with paths $P_m$, $P_n$ ($m \geq 2$, $n \geq 2$) and $P_2$ incident at different vertices,

or

5) $G$ is a cycle $C_5$ together with a path $P_n$, ($n \geq 2$) incident to a vertex of $C_5$,

or

6) $G$ is a cycle $C_5$ together with two paths $P_m$ and $P_n$ ($m \geq 1$, $n \geq 1$) adjoined at two consecutive vertices,

or

7) $G$ is a cycle of length $C_n$ ($n \geq 5$) together with two diagonal edges each joining a pair of vertices of length exactly two or together with two diagonal edges each joining a pair of vertices of length two and three which are adjacent,

or

8) $G$ is a cycle of length $C_n$ ($n \geq 6$) together with a diagonal edge joining a pair of vertices of length exactly 3.

**Proof.** Suppose $T_b(G)$ is 3-minimally nonouterplanar. Then $T_b(G)$ is planar.

We now consider the following cases.

**Case 1.** Assume $G$ is a tree. Then by Theorem F, $T(G)$ and $T_b(G)$ are isomorphic and hence by Theorem E, $G$ has maximum degree atmost 3 and every vertex of degree 3 is a cut vertex.

We consider subcases of case 1.

**Subcase 1.1.** Suppose $G$ has atleast two cut vertices of degree 3. Then $G$ has two subgraphs which are isomorphic to $K_{1,3}$. Then by Lemma 1, $i[T(K_{1,3})]=2$. Since $T(K_{1,3}) = T_b(K_{1,3})$, $i[T_b(K_{1,3})]=2$, since $T_b(K_{1,3}) \subseteq T_b(G)$, $i[T_b(G)]>3$, a contradiction.

**Subcase 1.2.** Suppose $G$ has a vertex $v$ lies on 3 blocks and each block has no end vertex. Then $G$ has a subgraph isomorphic to $S(K_{1,3})$. On planar embedding of $T_b(G)$, $i[T_b(S(K_{1,3}))] \geq 4$. Since $S(K_{1,3})$ is a subgraph of $G$, $i[T_b(G)] \geq 4$, a contradiction.

**Subcase 1.3.** Suppose $G$ has a vertex $v$ lies on 3-blocks in which at least one block has an end vertex of $G$. Then by condition (1) of Theorem A, $T_b(G)$ is a 2-minimally nonouterplanar, a contradiction.
Case 2. Assume G is not a tree.

We consider the following subcases of case (2).

Subcase 2.1. Suppose G has three cycles. Then we have the following subcases of subcase 2.1.

Subcase 2.1.1. Assume G has three cycles, in which two cycles are $C_3$ and other cycle is $C_n$ ($n \geq 4$). In $T_B(G)$, each $C_3$ gives $K_4$. Then $i(C_3) = 1$. For the cycle $C_n$ ($n \geq 4$), $i(C_n) \geq 2$. Hence, $i(T_B(G)) > 3$, a contradiction.

Subcase 2.1.2. Assume G has three cycles $C_3$ as blocks and these three blocks lie on a common cut vertex. Then in $T_B(G)$, each $C_3$ and corresponding block vertex forms $K_4$ as a subgraph. But in $T_B(G)$ the three block vertices of cycles are mutually adjacent. Further the edges joining the block vertices of $C_3$ increases the inner vertex number in a planar embedding. Hence, $i(T_B(G)) > 3$, a contradiction.

Subcase 2.2. Suppose G has four cycles $C_3$ as blocks. Then we have subcases of subcase 2.2.

Subcase 2.2.1. Assume G has four cycles $C_3$ as blocks, such that each two $C_3$ lie on a common cut vertex. The block vertices corresponds to cycles $C_3$ and corresponding vertices of cycles $C_3$ are adjacent in $T_B(G)$. Then each cycle $C_3$ forms $K_4$ as subgraphs in $T_B(G)$. Since, the block vertices are adjacent in $T_B(G)$. Then the edges joining these three vertices generates the increase in the inner vertex number. Thus, $i(T_B(G)) > 3$, a contradiction.

Subcase 2.2.2. Assume there exists a bridge between the cycles $C_3$. In $T_B(G)$ each cycle $C_3$ forms $K_4$ and bridges form triangles as subgraphs. In $T_B(G)$ the block vertices are also adjacent. Thus, $i(T_B(G)) > 3$, a contradiction.

Subcase 2.3. Suppose G has two cycles, $C_n$ ($n \geq 3$). Then we consider following subcases of subcase 2.3.

Subcase 2.3.1. Assume G has cycles $C_3$, as blocks $B_1$ and $B_2$. Then in $T_B(G)$ the block vertices $b_1$ and $b_2$ corresponds to $B_1$ and $B_2$, which are adjacent to every vertex of $B_1$ and $B_2$. Also block vertices $b_1$ and $b_2$ are adjacent in $T_B(G)$. Hence, in $K_4$ is an induced subgraph in $T_B(G)$. Clearly $i(T_B(G)) < 3$, a contradiction.

Subcase 2.3.2. Assume G has exactly two cycles $C_4$ as a block. Then in $T_B(G)$, each $C_4$ has $i(T_B(C_4)) = 2$. Since each $T_B(C_4)$ is an induced subgraph of $T_B(G)$. Then $i(T_B(G)) > 3$, a contradiction.

Subcase 2.3.3. Assume G has $C_3$ and $C_5$ as blocks. Then in planar embedding of $T_B(G)$, $i(T_B(C_3)) = 1$ and $i(T_B(C_5)) = 3$. Thus $i(T_B(G)) > 3$, a contradiction.

Subcase 2.3.4. Assume G has $C_4$ and $C_5$ as blocks. Then by subcase 2.3.2 and subcase 2.3.3, $i(T_B(G)) > 3$, a contradiction.

Subcase 2.4. Suppose G has a triangle
together with 3 paths $P_n (n \geq 2)$ incident at a unique vertex. Suppose each path is of length at most two. Then in depicting the $T_B(G)$ in any plane, $i[T_B(G)] > 3$, a contradiction.

**Subcase 2.5.** Suppose $G$ has a triangle together with paths $P_n (n \geq 2)$ incident at different vertices. Suppose three paths as $P_3$ which are incident at different vertices. Then in $T_B(G)$ each $P_3$ forms a triangle and a triangle of $G$ forms a subgraph as $K_4$. The edges $e_i \in E[T_B(G)]$ which are incident to the blocks vertices of paths of $G$ generates the inner vertex number of $T_B(G)$ as $i[T_B(G)] > 3$, a contradiction.

**Subcase 2.6.** Suppose $G$ is exactly one cycle $C_n (n \geq 3)$ together with a path $P_n (n \geq 2)$ incident at a vertex of a cycle. Then we consider following subcases of subcase 2.6.

**Subcase 2.6.1.** Assume $G$ is a $C_4$ as a block B, together with a path $P_n (n \geq 2)$ incident at a vertex. Then in $T_B(G)$ block vertex b corresponds to $C_4$ is adjacent to every vertex of B. Thus $i[T_B(C_4)] = 1$. The remaining blocks of $G$ give a triangle as subgraph in $T_B(G)$. Thus $i[T_B(P_n)] = 0$. Hence $i[T_B(G)] < 3$, a contradiction.

**Subcase 2.6.2.** Assume $G$ is a cycle $C_4$ together with a path $P_n (n \geq 2)$ incident at a vertex. Then in planar embedding of $T_B(G)$, it is easy to see that $i[T_B(G)] = 2$. Hence $i[T_B(G)] < 3$, a contradiction.

**Subcase 2.6.3.** Assume $G$ has a cycle $C_6$ as a block B, together with a path $P_n (n \geq 2)$ incident at a vertex. Then in $T_B(G)$ block vertex b which corresponds to $C_6$ is adjacent to every vertex of B. Thus $i[T_B(C_6)] > 3$. The remaining blocks of $G$ forms in $T_B(G)$ gives as outer planar induced subgraphs. Thus $i[T_B(P_n)] = 0$. Hence, $i[T_B(G)] > 3$, a contradiction.

**Subcase 2.7.** Suppose $G$ is a cycle $C_n (n \geq 4)$ together with two paths $P_m$ and $P_n (m \geq 1, n \geq 1)$ adjoined at two vertices of a $C_n$ ($n \geq 4$). Then we consider the following subcases of subcase 2.7.

**Subcase 2.7.1.** Assume $G$ is a $C_4$ together with two paths $P_m$ and $P_n (m \geq 1, n \geq 1)$ adjoined at two consecutive vertices. Then by Theorem A of condition (3), $i[T_B(G)] = 2$. Hence, $i[T_B(G)] < 3$, a contradiction.

**Subcase 2.7.2.** Assume $G$ is a $C_4$ as a block B together with two paths $P_m$ and $P_n (m \geq 1, n \geq 1)$ adjoined at two alternate vertices. Then in $T_B(G)$ the block vertex b which corresponds to B is adjacent to every vertex of B and adjacent block vertices of the paths in $T_B(G)$. In any planar embedding $i[T_B(G)] > 3$, a contradiction.

**Subcase 2.7.3.** Assume $G$ has a cycle $C_5$ together with two paths $P_m$ and $P_n (m \geq 1, n \geq 1)$ adjoined at two alternate vertices of $C_5$. Then it is easy to see that $i[T_B(G)] > 3$ in any planar embedding of $T_B(G)$, a contradiction.

**Subcase 2.8.** Suppose $G$ is a cycle of length $C_n (n \geq 5)$ together with two diagonal
edges joining a pair of vertices. Then we consider the following subcases of subcase 2.8.

**Subcase 2.8.1.** Assume two diagonal edges joining a pair of vertices of length exactly three. Let $B$ be a block of a $C_n$ ($n \geq 5$). Then in $T_B(G)$ block vertex $b$ corresponds to $C_n$ is adjacent to every vertex of $B$. In planar embedding of $T_B(G)$ in any plane we have $i[T_B(G)] = 5$. Hence, $i[T_B(G)] > 3$, a contradiction.

**Subcase 2.8.2.** Assume two diagonal edges joining a pair of vertices of length two and three from same vertex to two alternate vertices. Then in planar embedding of $T_B(G)$ it is easy to see that $i[T_B(G)] = 4$. Hence, $i[T_B(G)] > 3$, a contradiction.

**Subcase 2.8.3.** Assume two diagonal edges joining a pair of vertices of length exactly three from same vertex to two consecutive vertices. Then in planar embedding of $T_B(G)$ the four vertices of a cycle $C_n$ and block vertex $b$ corresponding to cycle $C_n$ are the only five inner vertices in $T_B(G)$. Thus $i[T_B(G)] = 5$. Hence, $i[T_B(G)] > 3$, a contradiction.

**Subcase 2.9.** Suppose $G$ is a cycle $C_n$ ($n \geq 6$) as a block together with a diagonal edge joining a pair of vertices. Then we have following subcases of subcase 2.9.

**Subcase 2.9.1.** Assume a diagonal edge joining a pair of vertices of length exactly three. Then $G$ contains one more cycle $C_n$ as a block. Clearly $i[T_B(G)] > 3$, a contradiction.

**Subcase 2.9.2.** Assume a diagonal edge joining a pair of vertices of length four. Then $i[T_B(G)] > 3$, a contradiction.

**Subcase 2.9.3.** Assume a diagonal edge joining a pair of vertices of length two. Then $i[T_B(G)] < 3$, a contradiction.

Conversely, suppose (1) holds. Then $G$ has exactly 3 cycles and each cycle is a block, such that every cut vertex of $G$ lies on at most two blocks and each triangle has at least one vertex of degree two. Then by Theorem D, $T_B(G)$ has exactly three wheels as blocks. We know that every wheel is a minimally nonouterplaner. Hence, $i[T_B(G)] = 3$.

Suppose (2) holds. Then $G$ has cycles $C_4$ and $C_3$ as blocks. We have following cases.

**Case 1.** Assume the cycle $C_4$ and $C_3$ have a vertex in common. Let cycle $C_4$ as vertices $\{v_1, v_2, v_3, v_4\}$ cycle $C_3$ as vertices $\{v_4, v_5, v_6\}$ in which $v_4$ is a cut vertex incident to both $C_3$ and $C_4$. Then in $T_B(G)$ the block vertex $b_1$ and corresponding vertices of cycle $C_4$ are mutually adjacent in the exterior region of a cycle $C_4$. Thus, vertices $v_1$ and $v_2$ of a cycle $C_4$ are only two inner vertices in $T_B(G)$. Similarly the block vertex $b_2$ and corresponding vertices of a cycle $C_3$ are mutually adjacent in the exterior region of a cycle $C_3$. Thus, vertex $v_6$ of a cycle $C_3$ is only one inner vertex in $T_B(G)$. Hence, $T_B(G)$ has three inner vertices. Thus $i[T_B(G)] = 3$. This proves (2).

**Case 2.** Assume the path $P_n$ ($n \geq 2$) in
between the cycles $C_4$ and $C_1$. Let cycle $C_4$ as vertices $\{v_1, v_2, v_3, v_4\}$ and cycle $C_3$ as vertices $\{v'_4, v_5, v_6\}$ and path $P_n$ as vertices $P_n = \{p_1, p_2, \ldots, p_{n-1}, p_n\}$ with out lose of generality let us assume that vertex $v_4$ of cycle $C_4$ is coincide with vertex $p_1$ of path $P_n$ and similarly vertex $v'_4$ of cycle $C_3$ coincide with vertex $P_n$ of path $P_n$. Let $b, b'$ and $b_1, b_2, \ldots, b_{n-1}$ are the block vertices corresponds to cycle $C_4$, $C_3$ and path $P_n$ respectively. Then in $T_B(G)$ the block vertex $b$ and corresponding vertices of cycle $C_4$ are embedded in a plane in such a way that they are mutually adjacent in the exterior region of a cycle $C_4$. Thus, vertices $v_1, v_2$ of a cycle $C_4$ are only two inner vertices in $T_B(G)$. Similarly block vertex $b'$ and corresponding vertices of cycle $C_3$ are mutually adjacent in the exterior region of a cycle $C_3$. This forms $K_4$ as a subgraph in $T_B(G)$. Thus, vertex $v_6$ is only one inner vertex in $T_B(G)$. The block vertices $b_1, b_2, \ldots, b_{n-1}$ are also adjacent to the corresponding vertices of path $P_n$. This forms triangles in $T_B(G)$. Here $T_B(G)$ is outer planar. Thus $T_B(G)$ has exactly three inner vertices as $v_1, v_2$, from cycle $C_4$ and $v_6$ from cycle $C_3$. Hence, $i[T_B(G)] = 3$. This proves (2).

Suppose (3) holds. Let $G$ is a triangle with vertices $\{v_1, v_2, v_3\}$, path $P_m$ with vertices $\{p_1, p_2, \ldots, p_{m-1}, p_m\}$, path $P_n$ with vertices $\{p'_1, p'_2, \ldots, p'_{n-1}, p'_n\}$, with out lose of generality let us assume that paths $P_m$ and $P_n$ are incident at vertex $v_1$ of a triangle. Then vertex $v_1$ of a triangle, vertex $p_1$ of path $P_m$ and vertex $p'_1$ of path $P_n$ are coincide. Clearly $(G-v_1)$ has disjoint paths. Then by Theorem C. $T_B(G-v_1)$ is outerplanar. Let $b$ be the block vertex corresponding to a triangle, $b'_1, b'_2, \ldots, b'_{m-1}$ are the block vertices corresponding to a path $P_m$ and $b''_1, b''_2, \ldots, b''_{n-1}$ are the block vertices corresponding to a path $P_n$. Then in $T_B(G)$ block vertex $b$ corresponding vertices of a triangle are mutually adjacent. Then $T_B(C_3)$ is isomorphic to a wheel. The block vertices $b'_1, b'_2, \ldots, b'_{m-1}$ and corresponding vertices $\{p_1, p_2, \ldots, p_{m-1}, p_m\}$ of a path $P_m$ forms $v_1p_1b'_1, p_2p_2b'_2, p_3p_3b'_3, \ldots, p_{m-1}p_mb'_m$ as triangles as an induced subgraphs in $T_B[G]$. Similarly path $P_n$ forms $v_1p'_2b''_1, p'_2p'_2b''_2, \ldots, p'_n(p'_n b''_{n-1}$ as triangles as an induced subgraphs in $T_B(G)$. The vertices, $v_2, v_3$ of a triangle and corresponding block vertex $b$ are only three inner vertices in $T_B(G)$ in any planar embedding. Hence $i[T_B(G)] = 3$. This proves (3).

Suppose (4) holds. Let $G$ is a triangle with vertices $\{v_1, v_2, v_3\}$, path $P_m$ with vertices $\{p_1, p_2, \ldots, p_{m-1}, p_m\}$, path $P_n$ with vertices $\{p'_1, p'_2, \ldots, p'_{n-1}, p'_n\}$ and path $P_2$ with vertices $\{x_1, x_2\}$. Without lose of generality let us assume that paths $P_m, P_n$ and $P_2$ are incident at $v_1, v_2, v_3$ respectively. Then the vertices $v_1 = p_1, v_2 = p'_1$ and $v_3 = x_1$. Let $b$ be the block vertex corresponding to a triangle, $b'_1, b'_2, \ldots, b'_{m-1}$ are the block vertices corresponding to a path $P_m$, $b''_1, b''_2, \ldots, b''_{n-1}$ are the block vertices corresponding to a path $P_n$ and $b'$ is a block vertex corresponding to a path $P_2$. Then in $T_B(G)$ block vertex $b$ corresponding vertices of a triangle are mutually adjacent. Then
\(T_B(C_3)\) is isomorphic to a wheel. The block vertices \(b'_1, b'_2, \ldots, b'_{m-1}\) and corresponding vertices \(\{p_1, p_2, \ldots, p_{m-1}, p_m\}\) of a path \(P_m\) forms \(v_1p_2b'_1, p_2p_3b'_2, p_3p_4b'_3, \ldots, p_{m-1}p_mb'_{m-1}\) as triangles as an induced subgraphs in \(T_B(G)\). Similarly path \(P_n\) forms \(v_2p'_2b''_1, p_2p'_3b''_2, \ldots, p'_{n-1}p'_nb''_{n-1}\) as triangles as an induced subgraphs in \(T_B(G)\). The vertex \(v_3\) of a triangle, vertex \(x_2\) of a path \(P_2\) and corresponding block vertex \(b'\) of a path \(P_2\) are exactly three inner vertices in \(T_B(G)\) in any planar embedding.

Hence, \(i[T_B(G)]=3\).

This proves (4).

Suppose (5) holds. Let \(G\) is a cycle \(C_5\) as a block \(B\), with vertices \(\{v_1, v_2, v_3, v_4, v_5\}\) and path \(P_n\) with vertices \(\{p_1, p_2, \ldots, p_n\}\). With out lose of generality let us assume that path \(P_n\) is incident at vertex \(v_1\) of a cycle \(C_5\). Thus vertex \(v_1=p_1\). Clearly \((G-v_1)\) has disjoint paths. Then by Theorem C, \(T_B(G-v_1)\) is outer planar. Let \(b\) be the block vertex corresponding to a cycle \(C_5\), \(b'_1, b'_2, \ldots, b'_{m-1}\) are the block vertices corresponding to a path \(P_m\) and \(b''_1, b''_2, \ldots, b''_{n-1}\) are the block vertices corresponding to a path \(P_n\). Then in \(T_B(G)\) block vertex \(b\) corresponding vertices of cycle \(C_5\) are mutually adjacent. Then \(T_B(C_5)\) is isomorphic to a wheel.

The block vertices \(b'_1, b'_2, \ldots, b'_{m-1}\) and corresponding vertices \(\{p_1, p_2, \ldots, p_m\}\) of a path \(P_m\) forms \(v_1p_2b'_1, p_2p_3b'_2, p_3p_4b'_3, \ldots, p_{m-1}p_mb'_{m-1}\) triangles as an induced subgraphs in \(T_B(G)\). Similarly path \(P_n\) forms \(v_2p'_2b''_1, p'_2p'_3b''_2, \ldots, p'_{n-1}p'_nb''_{n-1}\) as triangles as an induced subgraphs in \(T_B(G)\). The vertices \(v_2, v_3, v_4\) of a cycle \(C_5\) are only three inner vertices in \(T_B(G)\) in any planar embedding. Hence, \(i[T_B(G)]=3\). This proves (6).

Assume that (7) holds \(G\) has a cycle
$C_n (n \geq 5)$ as a block together with two diagonal edges.

We have the following cases.

Case 1. Assume a cycle $C_6$ with vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ together with two diagonal edges adjoined at $v_1$, $v_3$, and $v_4$, $v_6$. Clearly $(G-v_1)$ is a path. Let $b$ be a block vertex corresponding to a cycle $C_6$. Then in $T_B(G)$ block vertex $b$ is adjacent to every vertex of a cycle $C_5$. This forms a wheel in $T_B(G)$. Thus, vertices $v_2$, $v_3$ of a cycle $C_5$ and block vertex $b$ are only three inner vertices in $T_B(G)$. Hence, $i[T_B(G)] = 3$. This proves (7).

Suppose (8) holds. Then $G$ has a cycle $C_n (n \geq 6)$ with vertices $v_1, v_2, v_3, v_4, \ldots v_n$ such that $e=v_3v_n$ is an edge joining the two distinct vertices of $C_n$. Then $G$ has two cycles $v_1, v_2, v_3, v_n, v_1$ and $v_3, v_4, \ldots v_n, v_3$. Since $G$ is a block let $b$ be a block vertex of $G$. In $T_B(G)$, the block vertex $b$ and all vertices of $C_n (n \geq 6)$ are adjacent. Further, in planar embedding of $T_B(G)$, the edge $e=v_3v_n$ drawn in such a way that the vertices $v_1$, $v_2$, and $b$ lie in the interior region of $G$, which gives $i[T_B(G)] = 3$.

This completes the proof.

References


